131. *On Normal Approximate Spectrum.* VI

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1. Introduction. For a unital C^*-algebra $\mathfrak{A}$, the connectedness of the set $G[\mathfrak{A}]$ of all regular members of $\mathfrak{A}$ is discussed in several occasions: In an early stage, Kakutani observed in [14; pp. 280–281], $G[\mathfrak{A}]$ is connected if $\mathfrak{A}$ is the algebra $\mathfrak{B}(\mathfrak{H})$ of all operators acting on a Hilbert space $\mathfrak{H}$. Kuiper [13] proved that the homotopy group $\pi_m(G[\mathfrak{A}])$ vanishes for all m if $\mathfrak{A}=\mathfrak{B}(\mathfrak{H})$. Breuer [1] generalized Kuiper's theorem for every semifinite properly infinite factor. However, if $\mathfrak{A}$ is not large, then the situation changes. Kakutani pointed out in [14; p. 294], the set of all regular elements of the algebra $C(S^1)$ of all continuous functions on the unit circle S^1 is not connected: $G[C(S^1)]$ has infinitely many components each of which contains one of

$$e_n(s)=e^{2\pi ins} \qquad (n=0, \pm 1, \pm 2, \cdots). \tag{1}$$

In the present note, the connectedness of $G[\mathfrak{A}]$ for a general C^*-algebra $\mathfrak{A}$ is considered in § 2, where some theorems of Cordes and Labrousse [6] are given alternative proofs, and they are combined with a theorem of Royden [15]. In § 3, a unital C^*-algebra generated by an operator will be discussed; theorems on the algebraic theory of Fredholm operators, discussed by Breuer-Cordes [2] and Coburn-Lebow [4], are applied, and some elementary properties of the index are proved. In § 4, the unital C^*-algebra generated by the unilateral shift is discussed to illustrate these considerations. In §§ 3–4, the normal approximate spectrum of the generator plays a central role.

2. Connectedness. A member A of $G[\mathfrak{A}]$ of a unital C^*-algebra $\mathfrak{A}$ is *homotopic* (in $G[\mathfrak{A}]$) with $B\in G[\mathfrak{A}]$ if there is a continuous way $A_t(0\leqq t\leqq 1)$ in $G[\mathfrak{A}]$ with $A_0=A$ and $A_1=B$.

The following two theorems are obtained in [6] with somewhat different proofs:

Theorem 1 (Cordes-Labrousse). *If $H\in\mathfrak{A}$ is an invertible and positive element, then H is homotopic with* 1.

Define

$$H_t=t+(1-t)H \qquad (0\leqq t\leqq 1). \tag{2}$$

Then H_t is positive and invertible by the Gelfand representation (of the unital C^*-algebra generated by H). H_t is continuous in t with $H_0=H$ and $H_1=1$; hence H is homotopic with 1.

Theorem 2 (Cordes-Labrousse). *If $A\in G[\mathfrak{A}]$ and*