127. *On Fourier Coefficients of Certain Cusp Forms*

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§1. Congruences on certain bases of $S_k(\Gamma)$.

We shall denote by Γ the group $SL(2, \boldsymbol{Z})$. The set of integral automorphic forms (cusp forms) of weight k (k being a positive integer) with respect to Γ forms a vector space $G_k(\Gamma)$ ($S_k(\Gamma)$) over the complex number field $\boldsymbol{C}$, whose dimension is known to be (cf. [7] p. 48):

$$\dim G_k(\Gamma)=\begin{cases}[k/12] & (k\equiv 2 \pmod{12}),\\ [k/12]+1 & (k\not\equiv 2 \pmod{12}),\end{cases}$$

$$\dim S_k(\Gamma)=\begin{cases}0 & (k=2),\\ [k/12] & (k\not\equiv 2 \pmod{12}),\\ [k/12]-1 & (k>2,\ k\equiv 2 \pmod{12}).\end{cases}$$

Any element $\varphi(\tau)$ ($\tau\in\boldsymbol{C}$, $\operatorname{Im}\tau>0$) of $G_k(\Gamma)$ admits a Fourier expansion in $q=e^{2\pi i\tau}$:

$$\varphi(\tau)=\sum_{n=0}^{\infty}\alpha(n)q^n;$$

we have $\varphi(\tau)\in S_k(\Gamma)$ if and only if $\alpha(0)=0$.

Using Eisenstein series, one can obtain bases of $S_k(\Gamma)$ as follows. Following the notation in [9], we put

$$\begin{aligned}E_k(\tau)&=\frac{1}{2}\sum_{\substack{l,m\in\boldsymbol{Z}\\(l,m)\neq(0,0)}}(l\tau+m)^{-k} \qquad (k=4,6,8,\cdots)\\ &=\zeta(k)+\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\\ &=-\frac{(2\pi i)^kB_k}{2\cdot k!}+\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, B_k is the k-th Bernoulli number and

$$\sigma_g(n)=\sum_{\substack{t\mid n\\t>0}}t^g \qquad (g=0,1,2,3,\cdots).$$

We put further

$$E_k^*(\tau)=1-\frac{2\cdot k}{B_k}\sum\sigma_{k-1}(n)q^n \qquad (k=4,6,8,\cdots), \tag{1}$$

so that

$$E_k(\tau)=\zeta(k)\cdot E_k^*(\tau).$$

Then the well-known cusp form $\Delta(\tau)$ of weight 12 under the name of Ramanujan's function is written in the form: