158. *A Note on Character Sums*

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(Comm. by Kunihiko KODAIRA, M. J. A., Nov. 12, 1973)

§ 1. Introduction.

This is a continuation of the previous work (cf. [2]). We are concerned with the estimate of $\sum_{n\le X}\chi(n)$, where χ is a primitive character mod q. In [2] we showed

$$|\sum_{n\le X}\chi(n)|\ll_q\sqrt{X}\,q^{1/6},$$

where $\ll_q$ depends on prime factors of q. Here we will improve the dependence of $\ll_q$ on the prime factors of q. Hereafter implicit constants in $\ll$ are absolute. We will prove

Theorem. *Let χ be a primitive character* mod q. *Then for* $X\le q^{2/3}$.

$$|\sum_{n\le X}\chi(n)|\ll\sqrt{X}\,q^{1/6}B(q)$$

with

$$B(q)=\min_{q=q_1q_2}\left\{q_1^{1/3}(\log q_1)^{\delta_2}\left[\left(\frac{1}{3}\log(q_2q_1^2)\right)^{R_2/2}A_2^{1/3}q_1^{1/3}\Big/\left(\prod_{p_i|q_2}\log p_i\right)\right.\right.$$
$$\left.\left.+(\log q_2)^{1/2}A_2^{1/6}\left(\prod_{p_i|q_2}p_i\Big/(p_i-1)\right)^{1/2}\right]^{1-\delta_2}\right\}$$

where

1) $\min_{q=q_1q_2}$ *is taken over all decomposition of q into q_1q_2 such that if* $q_2=\prod_{i=1}^{R_2}p_i^{r_i}$, *then $p_i^{r_i}\|q$ and $r_i>r_0$, where r_0 is* 32, *say.*

2) $A_2=\prod_{p_i|q_2}p_i^{k_i}$, *where $K_i=0,2,1$ according as $r_i\equiv 0,1,2$* (mod 3) *and $p_i^{r_i}\|q_2$.*

3) $\delta_2=\begin{cases}1 & \text{if } q_2=1\\ 0 & \text{if } q_2\ne 1.\end{cases}$

§ 2. Proof of Theorem.

Let $q=q_1q_2$ and $q_2=\prod_{i=1}^{R_2}p_i^{r_i}$ with $p_i^{r_i}\|q$ and $r_i>r_0$. Let s_i be the least natural number larger than or equal to $r_i/3$. Write $d=\prod_{i=1}^{R_2}p_i^{s_i}$ and $k_i=0,2,1$ according as $r_i\equiv 0,1,2$ (mod 3). We have by definition $r_i+k_i=3s_i$. Let $A_2=\prod_{i=1}^{R_2}p_i^{k_i}$. If $X\le d$, the theorem comes from a trivial estimate. (For $|\sum_{n\le X}\chi(n)|\le\sqrt{X}=\sqrt{X}\sqrt{X}\le\sqrt{X}d^{1/2}=\sqrt{X}\,q_2^{1/6}A_2^{1/6}$.) Hence we assume $d\le X\le q^{2/3}$. We see that

$$\left|\sum_{n\le X}\chi(n)\right|\le\sum_{\mu=0}^{\mu_0}\left|\sum_{N\le n\le N'}\chi(n)\right|,$$

where $N=2^\mu d$, $N'\le 2N$, $\mu=0,1,\cdots,\mu_0$ and $2^{\mu_0}d\le X\le 2^{\mu_0+1}d$. Hence the problem is reduced to the estimate of the sums of the type $\sum_{N\le n\le N'}\chi(n)$ under $d\le N\le q^{2/3}$, $N'\le 2N$.