30. A Necessary Condition for the Well-Posedness of the Cauchy Problem for a Certain Class of Evolution Equations

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§ 1. **Introduction.** We consider the Cauchy problem for an evolution equation

$$(*)\qquad \begin{cases}(\partial_t - i\partial_x^2 - b(x,t)\partial_x)u(x,t)=0, & (x,t)\in \mathbf{R}^1\times[0,T],\\ u(x,0)=u_0(x),\end{cases}$$

where

$$b(x,t)\in \mathcal{E}_t^0(\mathcal{B}^\infty),\quad u_0(x)\in \mathcal{D}_{L^2}^\infty,\quad \partial_t=\frac{\partial}{\partial t},\quad \partial_x=\frac{\partial}{\partial x}.$$

Under what conditions is the Cauchy problem $(*)$ well posed?

In the case where $b(x,t)$ is constant, Hadamard's condition shows that the necessary and sufficient condition for the Cauchy problem $(*)$ to be well posed is that the coefficient b is a real number (see Theorem 5.3 in S. Mizohata [2]). In the case where $b(x,t)$ is a real-valued function, it is easy to see that the Cauchy problem $(*)$ is well posed in $\mathcal{D}_{L^2}^\infty$. In the case where $\mathcal{I}m\, b(x,t)\not\equiv 0$, as we shall see below, the situation is much more delicate. In order to make this situation clear, we assume that $b(x,t)$ is a function depending only on x, denote it by $b(x)$:

$$(**)\qquad \begin{cases}(\partial_t - i\partial_x^2 - b(x)\partial_x)u(x,t)=0 & (x,t)\in \mathbf{R}^1\times[0,T],\\ u(x,0)=u_0(x).\end{cases}$$

As we mentioned above, if we fix x_0 such that $\mathcal{I}m\, b(x_0)\neq 0$, then the Cauchy problem for the tangential operator (i.e. operator freezing the coefficients) $\partial_t - i\partial_x^2 - b(x_0)\partial_x$ is not well posed in $\mathcal{D}_{L^2}^\infty$. But in the case where the coefficients depend on x, the situation is different. The following assertion holds:

Assume that $\mathcal{I}m\, b(x)$ belongs to $L^1(\mathbf{R}^1)\cap\mathcal{B}^\infty$. Then the Cauchy problem $(**)$ is well posed in $\mathcal{D}_{L^2}^\infty$.

To see this, it is sufficient to note that the linear mapping

$$\mathcal{E}_t^1(\mathcal{D}_{L^2}^\infty)\ni u(x,t)\to v(x,t)=u(x,t)\exp\left(\frac{1}{2}\int_{-\infty}^x \mathcal{I}m\, b(y)dy\right)\in \mathcal{E}_t^1(\mathcal{D}_{L^2}^\infty)$$

is one-to-one, onto, continuous and that $v(x,t)$ satisfies the equation

$$(***)\qquad \begin{cases}(\partial_t - i\partial_x^2 - \mathcal{R}e\, b(x)\partial_x + c(x))v(x,t)=0,\\ v(x,0)=u_0(x)\exp\left(\frac{1}{2}\int_{-\infty}^x \mathcal{I}m\, b(y)dy\right),\end{cases}$$