31. *On Some Noncoercive Boundary Value Problems for the Laplacian*

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1. Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^n$ with boundary Γ of class C^∞. $\bar{\Omega}=\Omega\cup\Gamma$ is a C^∞-manifold with boundary. Let a, b and c be real valued C^∞-functions on Γ, let $\boldsymbol{n}$ be the unit exterior normal to Γ and let α and β be real C^∞-vector fields on Γ.

We shall consider the following boundary value problem: For given functions f defined on Ω and ϕ defined on Γ find u in Ω such that

$$(*)\qquad \begin{cases}(\lambda-\Delta)u=f & \text{in } \Omega,\\ \mathscr{B}u\equiv a\dfrac{\partial u}{\partial \boldsymbol{n}}+(\alpha+i\beta)u+(b+ic)u=\phi & \text{on } \Gamma.\end{cases}$$

Here $\lambda\geqq 0$ and $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$. The problem $(*)$ in the case that $\beta(x)\equiv 0$ on Γ, i.e., the *oblique* derivative problem was investigated by many authors (cf. [2], [6], [7], [8]), but the problem $(*)$ in the case that $\beta(x)\not\equiv 0$ on Γ was treated by a few authors, e.g., Vaĭnberg and Grušin [12] (see also [5]), whose results we shall first describe briefly. For each real s, we shall denote by $H^s(\Omega)$ (resp. $H^s(\Gamma)$) the Sobolev space on Ω (resp. Γ) of order s and by $\|\ \|_s$ (resp. $|\ |_s$) its norm.

If $a(x)>|\beta(x)|$ on Γ where $|\beta(x)|$ is the length of the tangent vector $\beta(x)$, then the problem $(*)$ is *coercive* and the following results are valid for all $s>3/2$ (cf. [9]):

i) For every solution $u\in H^t(\Omega)$ of $(*)$ with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-3/2}(\Gamma)$ we have $u\in H^s(\Omega)$ and an *a priori* estimate:

$$(1)\qquad \|u\|_s\leqq C_1(\|f\|_{s-2}+|\phi|_{s-3/2}+\|u\|_t)$$

where $t<s$ and $C_1>0$ is a constant depending only on λ, s and t.

ii) If $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-3/2}(\Gamma)$ and (f,ϕ) is orthogonal to some finite dimensional subspace of $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, then there is a solution $u\in H^s(\Omega)$ of $(*)$.

iii) If $\lambda>0$ is sufficiently large, then we can omit $\|u\|_t$ in the right hand side of (1) and for every $f\in H^{s-2}(\Omega)$ and every $\phi\in H^{s-3/2}(\Gamma)$ there is a unique solution $u\in H^s(\Omega)$ of $(*)$.

If $a(x)\geqq|\beta(x)|$ on Γ and $a(x)=|\beta(x)|$ holds at some points of Γ, then the problem $(*)$ is *noncoercive*. Vaĭnberg and Grušin [12] treated the problem $(*)$ in the case that $n=2$, $a(x)\equiv 1$, $\alpha(x)\equiv 0$, $|\beta(x)|\equiv 1$ on Γ. Under the assumption that $b(x)+ic(x)\neq 0$ on Γ, they proved smoothness, an *a priori* estimate and existence theorems for the solutions of