70. *On the Structure of the Graded C-Algebras of Theta Functions*

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Introduction. Let $(X, \mathcal{L})$ be a polarized abelian variety of dimension n over the complex number field $\boldsymbol{C}$ defined by an abelian variety X and an ample invertible sheaf $\mathcal{L}$ on X. The purpose of this note is to give some information on the graded $\boldsymbol{C}$-algebra $\mathcal{A}(X, \mathcal{L}) = \bigoplus_{\alpha=0}^{\infty} H^0(X, \mathcal{L}^\alpha)$. We understand, by the Siegel space $\mathcal{H}_n$ of degree n, the set of all complex symmetric matrices with positive imaginary part. We have, for $z \in \mathcal{H}_n$ and a square matrix e of size n with coefficients in $\boldsymbol{Z}$ and $\det e \neq 0$, a naturally polarized complex torus $\boldsymbol{C}^n/\langle z, e\rangle$ where $\langle z, e\rangle$ is the lattice subgroup of $\boldsymbol{C}^n$ generated by the column vectors of the $(n \times 2n)$-matrix (z, e). After a suitable choice of (z, e), $\boldsymbol{C}^n/\langle z, e\rangle$ is isomorphic to $(X, \mathcal{L})$ and the investigation of $\mathcal{A}(X, \mathcal{L})$ is reduced to that of the graded $\boldsymbol{C}$-algebra consisting of theta functions on $\boldsymbol{C}^n$ relative to $\langle z, e\rangle$ of certain types.

Besides the above notations $\mathcal{H}_n$, e, etc., we indicate some definitions and notations which are used throughout this note. $\boldsymbol{C}^n$ etc. denote the set of column n-vectors with components in $\boldsymbol{C}$ etc. $\boldsymbol{Z}_+$ and $\boldsymbol{N}$ denote the set of all positive integers and $\boldsymbol{Z}_+ \cup \{0\}$, respectively. For $\alpha, \beta \in \boldsymbol{Z}_+$, g.c.d. (α, β) is the greatest common divisor of α and β. In general for e as above (resp. $\alpha \in \boldsymbol{Z}_+$), U_e (resp. U_α) denotes a complete set of representatives of ${}^t e^{-1}\boldsymbol{Z}^n$ (resp. $\alpha^{-1}\boldsymbol{Z}^n$) mod $\boldsymbol{Z}^n$, which is once for all fixed through in a discussion. When we write $k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \boldsymbol{R}^{2n}$, k_1 and k_2 are its upper and lower halves in $\boldsymbol{R}^n$. We put $\boldsymbol{e}(t) = \exp(2\pi\sqrt{-1}t)$. $\theta[k](z \mid x)$ is a holomorphic function on $(z, x) \in \mathcal{H}_n \times \boldsymbol{C}^n$, which is defined by

$$\theta[k](z \mid x) = \sum_{r \in \boldsymbol{Z}^n} \boldsymbol{e}\left\{\frac{1}{2}{}^t(r+k_1)z(r+k_1) + {}^t(r+k_1)(x+k_2)\right\}.$$

For $\alpha \in \boldsymbol{N}$, $z \in \mathcal{H}_n$ and $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \boldsymbol{R}^{2n}$, an entire function $\xi(x)$ not identically zero on $\boldsymbol{C}^n$ is called a theta function of type $((z, e), m)_\alpha$, if the period relation, for any $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \boldsymbol{Z}^{2n}$,

$$\xi\left(x + (z, e)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\right) \cdot \xi(x)^{-1} = \boldsymbol{e}\left\{\alpha\left(-{}^t s_1 x - \frac{1}{2}{}^t s_1 z s_1 + ({}^t s_1, {}^t s_2)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right)\right\}$$