85. *On the Korteweg-de Vries Equation*

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1. Introduction. Gardner, Greene, Kruskal and Miura [1] discovered that the discrete eigenvalues of one-dimensional Schrödinger operators $L(t)=d^2/dx^2+u(x,t)$ are constants in t while the potential $u(x,t)$ varies according to the Korteweg de Vries (KdV) equation:

(1.1) $$u_t+6uu_x+u_{xxx}=0, \qquad -\infty<x,t<+\infty,$$

where the subscripts x,t denote partial differentiations. From this they gave a method of constructing solutions of the KdV equation by means of the inverse scattering problem for $L(t)$. Lax [2] presented a general principle for a family of selfadjoint operators $L(t)$ to be unitary equivalent. Applying this principle he gave another proof of the invariance of the eigenvalues and derived an infinite family of equations (higher order KdV equations) that leave the eigenvalues of $L(t)$ invariant in time. Menikoff [3] gave another criterion for the invariance of the eigenvalues of $L(t)$, which works in the case when $u(x,t)\to-\infty$ as $|x|\to+\infty$. His basic idea is to associate the eigenvalue problem for $L(t)$ the following boundary value problem of parabolic type:

$$G_s=G_{xx}+u(x,t)G,$$
$$\lim_{s\to 0} G(x,y,s\,;t)=\delta(x-y),$$
$$\lim_{|x|\to+\infty} G(x,y,s\,;t)=0.$$

In the class of periodic functions the author [4] has given another characterization of an infinite family of higher order KdV equations and presented a constructive method of deriving them.

In this paper, our purpose is to derive an infinite family of the (higher order) KdV equations and their conserved densities by means of an improved version of the Menikoff's method. Here we shall consider the periodic boundary value problems.

2. Invariance of the eigenvalues and derivation of the infinite family of (higher order) KdV equations. Let $u(x,t)$ be infinitely differentiable real functions of x and t in $R^1\times R^1$ and periodic with respect to x with period 1. Consider the eigenvalue problem with t considered as a parameter:

(2.1) $$\begin{cases} L(t)\varphi=\varphi_{xx}+u(x,t)\varphi=-\lambda\varphi, \\ \varphi(x,t)=\varphi(x+1,t), \\ \varphi_x(x,t)=\varphi_x(x+1,t). \end{cases}$$