6. On Polythetic Groups

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§ 1. Let G be a locally compact abelian (LCA) group and Z be the additive group of integers. We say G is polythetic if it has a dense subgroup which is a homomorphic image of Z^n. In other words G is to contain n elements $x_1, \cdots, x_n$ such that the subgroup

$$\{m_1x_1 + \cdots + m_nx_n\,;\,(m_1, \cdots, m_n) \in Z^n\}$$

is dense in G. We call such elements $x_1, \cdots, x_n$ '*generators of* G'.

In the case $n=1$, G is called *monothetic* and for compact monothetic groups their characterization is stated in terms of their duals by Halmos and Samelson [1]. In this paper we have characterization of LCA polythetic groups by their structures and the smallest numbers of their generators. For the terminologies and notations in this note, see Rudin [2].

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§ 2. For a LCA polythetic group G let $A(G)$ be the set of integers $n>0$ such that there exists a homomorphic image of Z^n which is dense in G. Clearly $A(G)$ has the smallest element, which we denote by $s(G)$.

Now we state the characterization of compact polythetic groups.

The annihilator Λ of a closed subgroup H of G is the set of all $\gamma \in \Gamma$ (the dual group of G) such that $(x, \gamma)=1$ for all $x \in H$. Λ forms a closed subgroup of Γ.

Lemma 1. *For $i=1, \cdots, n$, let H_i be the closure of the subgroup generated by $x_i \in G$, Λ_i be its annihilator, and let H be the subgroup of G generated by $x_1, \cdots, x_n$. H is dense in G if and only if $\bigcap_{i=1}^{n} \Lambda_i = \{0\}$.*

We denote by T the multiplicative group of all complex numbers of absolute value 1 with the usual topology (or equivalently the additive group of real numbers mod 2π) and by T_d the same group with the discrete topology.

Theorem 1. *Let G be a compact abelian group. G is polythetic if and only if Γ is isomorphic to a subgroup of T_d^n.*

Proof. If G is polythetic, G has generators $x_1, \cdots, x_n$. Since the natural mapping α of T_d^n onto T^n is an algebraic isomorphism, the mapping $\gamma \to \alpha^{-1}((x_1, \gamma), \cdots, (x_n, \gamma))$ is an isomorphism of Γ into T_d^n, be-