25. *Putcha's Problem on Maximal Cancellative Subsemigroups*

By Takayuki TAMURA
University of California, Davis, California, 95616, U. S. A.

(Comm. by Kenjiro SHODA, M. J. A., March 12, 1976)

1. Introduction. Let S be a commutative archimedean semigroup without idempotent ([1], [3], [5]). M. S. Putcha asked the following question in his recent paper [4].

Is every maximal cancellative subsemigroup of S necessarily archimedean?

In this paper the author negatively answers this question by exhibiting a counter example and discusses a further problem. Throughout this paper, Z denotes the set of integers, Z_+ the set of positive integers and Z_+^0 the set of nonnegative integers. Let S be a commutative semigroup and let a be any element of S. Define ρ_a on S by

$$x\rho_a y \quad \text{if and only if} \quad a^m x = a^n y \quad \text{for some} \quad m, n \in Z_+.$$

Then ρ_a is a congruence on S, and if S is a commutative archimedean semigroup without idempotent, then S/ρ_a is a group [5], [6]. Let $G_a = S/\rho_a$. G_a is called the *structure group* of S with respect to a. A commutative semigroup S is called power joined if, for any $a, b \in S$, there are $m, n \in Z_+$ such that $a^m = b^n$.

Putcha's question is affirmative if G_a is torsion. It is more strongly stated as follows:

Proposition 1.1. *Let S be a commutative archimedean semigroup without idempotent. If a structure group of S is torsion, then every subsemigroup of S is archimedean.*

Proof. According to [2], S is power joined if and only if G_a is torsion for some $a \in S$, equivalently for all $a \in S$. Every subsemigroup of S is power joined, hence archimedean.

Accordingly Putcha's question is interesting only in the case G_a is not torsion.

2. Counter example. Let G be the free abelian group of rank $r \geqq 2$, where r may be infinite, but we assume $2 \leqq r \leqq \aleph_0$ for our convenience. However this restriction will be easily removed later. Every element λ of G will be expressed by

$$\lambda = (\lambda_1, \cdots, \lambda_i, \cdots) \quad \text{or} \quad (\lambda_i)$$

where $\lambda_i \in Z$ for all $i \in Z_+$, but if $r = \aleph_0$, only a finite number of λ_i's are not zero. The operation is defined by $(\lambda_i) + (\mu_i) = (\lambda_i + \mu_i)$ and the identity is $\mathbf{0} = (0)$. Define subsemigroups H and E of G by