73. *Group Rings of Metacyclic p-Groups*

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Let K be a field with characteristic $p>0$, P a finite p-group and KP a group ring of P over K. Recently W. Müller [6] proved that every left ideal of KP is generated by at most 2 elements if $p=2$ and P is either a dihedral group, a semi-dihedral group or a generalized quaternion group of order 2^{n+1}. These groups are metacyclic 2-groups. So in this paper we shall generalize the above result as follows: If P is a metacyclic p-group containing a cyclic normal subgroup Q and with a cyclic factor group P/Q, then every left (right) ideal of KP is generated by at most $|P/Q|$ elements. Further we shall show that there exists a metacyclic p-group P such that KP has a left (right) ideal whose minimal generators consist of $|P/Q|$ elements. By using our technique if P is a semi-direct product of Q by P/Q it is proved a relation among the nilpotency indices of the radicals of KP, KQ and $K(P/Q)$ which is similar in the case of a direct product of groups.

Let P be a metacyclic p-group containing a cyclic normal subgroup $Q=[b]$ of order p^n $(n\geqq 1)$ and with a cyclic factor group $P/Q=[aQ]$ of order p^m (cf. [1, §47]). Then there is an integer r such that $aba^{-1}=b^r$. Since $a^{p^m}\in Q$, $r^{p^m}\equiv 1 \pmod{p^n}$. Hence

$$(*)\qquad ba^i=a^i b^{r^{p^m-i}},\quad \text{for } i=0,\cdots,p^m-1.$$

We may put $a^{p^m}=b^{p^k}$, $(0\leqq k\leqq n)$. Put $B=KP$, $x=a-1$, $y=b-1$ in B and $[s,t]=x^{p^m-1-s}y^{p^n-1-t}$, for $s=0,\cdots,p^m-1$; $t=0,\cdots,p^n-1$. Then $C=\{[s,t]\,|\,0\leqq s\leqq p^m-1,\ 0\leqq t\leqq p^n-1\}$ forms a K-basis of B. Next we make C a totally ordered set by introducing in the following way: $[s,t]<[s',t']$ if and only if $t<t'$, or $t=t'$ and $s<s'$. Since each $u\in B\backslash\{0\}$ can be expressed uniquely in the form $u=\sum_{i=1}^{d}k_{iu}c_{iu}$, where $k_{iu}\in K\backslash\{0\}$, $c_{iu}\in C$, for $i=1,\cdots,d$ and $c_{1u}<c_{2u}<\cdots<c_{du}$, we can define a map $h: B\backslash\{0\}\to C$ such that $h(u)=c_{du}$. Put $\binom{i}{j}=0$ if $i<j$ or $j<0$.

At first we shall prove the following

Lemma. (a) $x[s,t]=[s-1,t]$, *for* $s=1,\cdots,p^m-1$; $t=0,\cdots,p^n-1$.

(b) $x[0,t]=0$, *for* $t=0,\cdots,p^k-1$. $x[0,t]=[p^m-1,t-p^k]$, *for* $t=p^k,p^k+1,\cdots,p^n-1$, if $k<n$.

(c) $y[p^m-1,t]=[p^m-1,t-1]$, *for* $t=1,\cdots,p^n-1$. $y[p^m-1,0]=0$.

(d) $h(y[s,t])=[s,t-1]$, *for* $s=0,\cdots,p^m-1$; $t=1,\cdots,p^n-1$. $y[s,0]=0$, *for* $s=0,\cdots,p^m-1$.