150. On the Jordan-Hölder Theorem

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Let $\{A_n, f_n\}$ be a family of groups A_n and homomorphisms $f_n: A_n \to A_{n-1}$, defined for all $n \in Z$ ($Z=\{0, \pm 1, \pm 2, \cdots\}$). If a sequence

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is exact, then we denote it by $(A_n : f_n)$ and we say $(A_n : f_n)$ to be *well defined.* Generalizations of Isomorphism Theorem and the Jordan-Hölder Theorem in group theory have been given in some papers (for example, [2] and [3]). The purpose of this note is also to give those theorems for a sequence $(A_n : f_n)$.

1. Isomorphism Theorem. In this section, let $(A_n : f_n)$ and $(B_n : g_n)$ be well defined. A *translation* $\{\alpha_n\}$ of $(A_n : f_n)$ into $(B_n : g_n)$ is the set of homomorphisms $\alpha_n : A_n \to B_n$ such that $\alpha_{n-1} f_n = g_n \alpha_n$ for all $n \in Z$. Moreover, if each α_n is an isomorphism, we say that $(A_n : f_n)$ is *isomorphic* to $(B_n : g_n)$. If for each $n \in Z$, B_n is a subgroup of A_n, i.e., $A_n \geqslant B_n$, and $f_n = g_n$ on B_n, then we denote $(B_n : g_n)$ by $(B_n : f_n)$. In this case, we call $(B_n : f_n)$ a *subsequence* of $(A_n : f_n)$ and write it in the notation: $(A_n : f_n) \geqslant (B_n : f_n)$. Moreover, if $A_n \triangleright B_n$ for all $n \in Z$, we call $(B_n : f_n)$ a *normal subsequence* of $(A_n : f_n)$ and write it in the notation: $(A_n : f_n) \triangleright (B_n : f_n)$.

It is easy to prove the following

Lemma 1. *Let $(A_n : f_n)$ be well defined. For each $n \in Z$, let M_n be a subgroup of A_n. Then $(M_n : f_n)$ is well defined iff $f_n(M_n) = f_n(A_n) \cap M_{n-1}$ for all $n \in Z$.*

By Lemma 1 and the same way as in proofs of [1, Lemma 2] and [1, Lemma 3], we can prove the following

Lemma 2. *Let $(A_n : f_n) \geqslant (P_n : f_n)$. For each $n \in Z$, let $A_n \geqslant M_n \triangleright P_n$. Then $(M_n : f_n)$ is well defined iff $(M_n / P_n : \bar{f}_n)$ is well defined where each $\bar{f}_n$ is a mapping which is naturally induced by f_n.*

Theorem 1. *Let $\{\alpha_n\} : (A_n : f_n) \to (B_n : g_n)$ be a translation. Then $(\alpha_n(A_n) : g_n)$ is well defined iff* $(\mathrm{Ker}\,(\alpha_n) : f_n)$ *is well defined. In this case, $(A_n / \mathrm{Ker}\,(\alpha_n) : \bar{f}_n)$ is also well defined and isomorphic to $(\alpha_n(A_n) : g_n)$, where for each $n \in Z$, $\bar{f}_n$ is a mapping which is naturally induced by f_n.*

Proof. The first assertion follows from routine arguments and the remainder follows from Lemma 2.