118. On a Generalization of Picard's Theorem.

By Masatsugu TSUJI.
Mathematical Institute, Tokyo Imperial University.

(Rec. July 15, 1926. Comm. by T. TAKAGI, M.I.A., Oct. 12, 1926.)

Landau has shown that, if $f(x)=a_0+a_1x+a_2x^2+\cdots\cdots$ is a transcendental integral function, where $a_0\neq 0, 1$, $a_1\neq 0$, then $f(x)$ takes the value 0 or 1 in a circle $|x|\leqq R(a_0, a_1)$, where R depends only on a_0 and a_1. The following theorem gives some more information about *the distribution of 0 and 1-points of $f(x)$ outside this circle.*

Theorem. Let $f(x)=a_0+a_1x+\cdots\cdots$ be an integral function, where $a_0\neq 0$ and $f(x_\nu)=0$, $(0<|x_1|<|x_2|<\cdots\to\infty)$ and a and b be given constants, then there exists a sequence of circles $|x|=R_\nu$, $(0<R_1<R_2\cdots\to\infty)$ such that in any ring region $R_\nu<|x|\leqq R_{\nu+1}$ $(R_0=0,\ \nu=0, 1\cdots)$, $f(x)$ takes the value a or b; the radii of circles $|x|=R_\nu$ $(\nu=1, 2\cdots)$ depending only on a_0, x_ν $(\nu=1, 2\cdots)$ and a, b.

The condition imposed on $f(x)$ requires only that it should vanish at x_ν; the multiplicity of zero is arbitrary, and $f(x)$ may vanish at other points than x_ν.

Lemma. Under the condition of the theorem, when a circle $|x|=R$ is given, we can find a second circle $|x|=R'$ $(R<R')$, so that $f(x)$ takes the value a or b in the ring region $R<|x|\leqq R'$, where R' depends only on a_0, x_ν $(\nu=1, 2\cdots)$, a, b and R.

Suppose that the lemma is false, then we can find a sequence of circles $|x|=R_\nu$ $(R<R_1<R_2\cdots\to\infty)$ and functions $f_\nu(x)$ so that $f_\nu(x)$ does not take the values a and b in the ring region $R<|x|\leqq R_\nu$, where $f(x)$ satisfies the condition of our Theorem.

Since $f_1(x)$, $f_2(x)$, $\cdots\cdots$ do not take the values a and b in $R<|x|\leqq R_1$, they form a normal family, so that we can select a subsequence $f_{11}(x)$, $f_{12}(x)$,$\cdots\cdots$, which converge uniformly in $R<|x|<R_1$. Since $f_{12}(x)$, $f_{13}(x)$,$\cdots\cdots$ do not take the values a and b in the ring region $R<|x|\leqq R_2$, we can select a sub-sequence $f_{22}(x)$, $f_{23}(x)\cdots\cdots$ which converge uniformly in $R<|x|<R_2$, and so on. Thus we get a sequence $f_{11}(x)$, $f_{22}(x)$, $f_{33}(x)$,$\cdots\cdots$ which converge uniformly in $R<|x|<R'$, where R' is any large number such that in $R<|x|<R'$ there exists at least one x_ν.