PAPERS COMMUNICATED

156. *On the Singularity of the Functions Defined by Dirichlet's Series.*

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The object of this paper is to extend Vivanti's theorem and its generalizations to functions defined by Dirichlet's series.

1. Let $r_1, r_2, r_3, \dots$ be a sequence of real numbers such that

$$0 < r_1 < r_2 < r_3 < \dots\dots, \quad \frac{r_\nu}{\nu} \to \infty.$$

Then the integral function

$$G(z) = \prod_{\nu=1}^{\infty}\left(1 - \frac{z^2}{r_\nu^2}\right)^2 \tag{1.1}$$

is of order 1 and of minimal type. Let us next consider the Dirichlet's series:

$$D(s) = \sum_{\nu=1}^{\infty} c_{\lambda_\nu} e^{-\lambda_\nu s} \quad \left(\begin{array}{c} c_{\lambda_\nu} = a_{\lambda_\nu} + ib_{\lambda_\nu} \\ 0 \leqq \lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad \lambda_\nu \to \infty \end{array}\right). \tag{1.2}$$

Then we have

Lemma 1. *The Dirichlet's series*

$$H(s) = \sum_{\nu=1}^{\infty} c_{\lambda_\nu} G(\lambda_\nu) e^{-\lambda_\nu s} \tag{1.3}$$

and (1.2) *have the same convergence abscissa, when*

$$\varliminf_{\nu=\infty} (\lambda_\nu - \lambda_{\nu-1}), \quad \varliminf_{\varkappa,\nu=\infty} (r_\varkappa - \lambda_\nu) > 0. \tag{1.4}$$

After Dr. Cramér[1] (1.3) converges at least in the domain, where (1.2) is convergent. So it suffices to prove the converse. To this purpose we will first calculate the order of $G(\lambda_\mu)$.

Let n be an integer such that $r_n < \lambda_\mu < r_{n+1}$. By (1.4) we have then $r_\nu - r_{\nu-1} > h$, $r_\varkappa - \lambda_\nu > h$ for all ν and $\varkappa$. In general we can suppose that $h=1$.

Now[2]
$$\frac{1}{G(\lambda_\mu)} \leqq \prod_{\nu=1}^{n} \frac{1}{\left(\frac{\lambda_\mu}{r_\nu} - 1\right)^2} \prod_{\nu=n+1}^{\infty}\left(1 + \frac{\lambda_\mu^2}{(r_\nu + \lambda_\mu)(r_\nu - \lambda_\mu)}\right)^2$$

$$\leqq \frac{\lambda_\mu^{2n}}{(n!)^2} \prod_{\nu=n+1}^{\infty}\left(1 + \frac{\lambda_\mu^2 \varepsilon_\nu}{\nu(\nu - n)}\right)^2 \quad \left(\varepsilon_\nu = \frac{\nu}{r_\nu} < \varepsilon^2\right)$$

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1) Cramér, Arkiv för Math. **13** (1919).
2) See Carlson u. Landau, Göttinger Nachrichten, 1921.