35. *On the Theory of Meromorphic Functions.*

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(Rec. Feb. 6, 1929. Comm. by T. TAKAGI, M.I.A., Feb. 12, 1929.)

Let $w=f(z)$ be a meromorphic function in the whole finite z-plane. Now consider the function depending on $|z|=r$ and $f(z)$:

$$A(r,f)=\frac{1}{\pi}\int_0^r\int_0^{2\pi}\frac{|(\rho f'e^{i\theta})|^2}{(1+|f(\rho e^{i\theta})|^2)^2}\rho d\rho d\theta\,, \quad\ldots\ldots\ldots\ldots\ldots \quad (1)$$

which is the area of the domain mapped by $w=f(z)$ for $|z|\leqq r$, and projected on the Riemann sphere of radius $\frac{1}{2}$ touching the w-plane at the origin, divided by the whole area of the Riemann sphere; that is, a mean number of sheets of the Riemann surface of the inverse function of $f(z)$ in $|z|<r$.

By the identity, which holds in the domain where $f(z)$ is regular:

$$\frac{4|f'(z)|)^2}{(1+|f(z)|^2)^2}=\Delta\log(1+|f(z)|^2)\,,$$

and Green's transformation formula in the domain $|z|\leqq r$, excluding small circles about the poles of $f(z)$ in $|z|<r$:

$$\iint(u\Delta v-v\Delta u)\,d\sigma=\int\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds\,,$$

we obtain, putting $u\equiv 1$, $v\equiv\log(1+|f(z)|^2)$,

$$A(r,f)=\frac{1}{4\pi}\int_0^{2\pi}\frac{\partial\log(1+|f(re^{i\theta})|^2)}{\partial r}r d\theta+n(r,\infty)\,,$$

when $n(r,\infty)$ denotes the number of the poles of $f(z)$ in $|z|<r$.

Putting $\quad b(r,f)=\dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\frac{\partial\log(1+|f(re^{i\theta})|^2)}{\partial r}r d\theta\,,$

we have obtained the following theorems.

Theorem I. $\quad A(r,f)=b(r,f)+n(r,\infty)\,. \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$

Theorem II. $\quad A(r,f)$ *is a continuous positive increasing function of* r.

Dividing by r and integrating with respect to r the right hand side of (2) from $\varepsilon>0$ to r we have