152. Note on the System of the Orthogonal Functions.

By Tatsuo TAKAHASHI.

Mathematical Institute, Tohoku Imperial University, Sendai.

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Let $\{\varphi_i(s)\}$ be an orthogonal system of functions which are defined and squarely integrable in (0, 1), and $x(s)$ be a function defined in the same interval. The formal series

$$\sum_{i=1}^{\infty}\varphi_i(s)\int_0^1 x(s)\varphi_i(s)ds \tag{1}$$

is called the expansion of $x(s)$ by the system $\{\varphi_i(s)\}$.

Concerning the expansion (1), Haar had, in his Dissertation,[1] proved the following theorems.

I. *Let s_0 be a point in* (0, 1), *and put*

$$K_n(s_0, s)=\sum_{i=1}^{n}\varphi_i(s_0)\varphi_i(s)$$

and

$$\omega_n=\int_0^1 |K_n(s_0, s)|\,ds\,.$$

If $\{\omega_n\}$ is not bounded, then there exists a continuous function whose expansion diverges at $s=s_0$.

II. *If every continuous function is uniformly approximable by the system $\{\varphi_i(s)\}$ in* (0, 1) *and $\{\omega_n\}$ is bounded, then the expansion of every continuous function converges at $s=s_0$.*

In the present paper, we prove these theorems by using theorems in the theory of linear operations, and at the same time, prove the following theorem.

III. *If the hypothesis in* I *is satisfied, then the set of functions*

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1) A. Haar: Zur Theorie der orthogonalen Funktionensysteme, Math. Ann. **69** (1912).

Cf. H. Steinhaus: Sur les développments orthogonaux, Bull. de Acad. de Cracovie, Série A (1926).

Banach-Steinhaus: Sur les principes de la condénsation de singularités, Fund. Math., **9** (1927).

W. Orlicz: Einige Bemerkungen über die Divergenzpunktmengen von Orthogonalentwicklungen, Studia Math. **2** (1930).

W. Orlicz: Eine Bemerkungen über Divergenzphänomene von Orthogonalentwicklungen, ibid.

2) The theorem in Haar's paper is a little more precise than this, but essencially equivalent.