66. A Remark on an Integral Equation.

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Mr. Nagumo has proposed the problem to solve the integral equation

$$f(x)=\frac{1}{2}\int_{x-1}^{x+1}f(t)dt. \tag{1}$$

In this paper two types of solutions are found. The first is of exponential type and the second is that belonging to L^2-class.

Theorem 1. If $f(x)$ is a solution of (1) such that

$$f(x)=O(e^{A|x|}),$$

A being a positive number, $f(x)$ is of the form

$$A'x+B+\sum ae^{-u^*x},$$

where u^* is the non-zero root of the equation

$$1=\frac{e^u-e^{-u}}{2u}, \tag{2}$$

such that $|R(u^*)|<A$ and A', B, a are arbitrary constants.

Proof. If we put

$$K(x)=\frac{1}{2},\quad |x|\leqq 1; \qquad K(x)=0,\quad |x|>1,$$

then (1) becomes

$$f(x)=\int_{-\infty}^{\infty}K(x-t)f(t)dt.$$

Therefore we can apply the theory of Hopf and Wiener.[1] As easily be seen, (2) has the origin as only one double zero. Thus we get the theorem.

Theorem 2.[2] If $f(x)$ is a solution of (1) belonging to L^2-class in $(-\infty, \infty)$, then $f(x)$ is identically zero.

Proof. We have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)e^{-uxi}dx&=\frac{1}{2\sqrt{2\pi}}\int_{-A}^{A}e^{-uxi}dx\int_{x-1}^{x+1}f(t)dt\\&=\frac{1}{2\sqrt{2\pi}}\Bigg[\int_{-(A-1)}^{A-1}f(t)dt\int_{t-1}^{t+1}e^{-uxi}dx\\&\qquad+\int_{-A-1}^{-A+1}f(t)dt\int_{-A}^{t-1}e^{-uxi}dx+\int_{A-1}^{A+1}f(t)dt\int_{t-1}^{A}e^{-uxi}dx\Bigg]\end{aligned}$$

1) E. Hopf and N. Wiener: Sitzungsberichte der Preussischen Akademie, 1931. Cf. E. Hopf: ibid., 1928, and Paley-Wiener: Fourier transforms in the complex domain, 1934, Chapter IV.

2) Cf. Hardy-Titchmarsh: Proc. London Math. Soc., (2) **23** (1924) and **30** (1930).