50. The Composition of Permutable Functions and the Operational Calculus.

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1. It is well known that by the Pérès's transformation[1] the Volterra-Pérès's first composition of the functions which are permutable with a certain given function $f(x, y)$ of the first order[2] may be transformed into the calculus of the operators of closed cycle, i.e. of the linear translatable operators. The object of this note is to show that the theory may be also considered as a special case of the operational calculus under our formulation.[3] We will verify this by suitably defining the fundamental operator $\mathfrak{D}$.[4] In the following we may- and we shall assume that $f(x, y)$ is in the canonical form[5] in the domain $0 \leqq x \leqq y \leqq a$ $(a > 0)$.

2. The group of the fuctions $g(x, y)$[6] which are permutable with a given $f(x, y)$ can be generated by a Pérès's transformation,[7] i. e., by one of the transformations that maintain the property of composition: let us put

$$h(x, y) \equiv \frac{\partial^2 f(x, y)}{\partial x \partial y} \tag{1}$$

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1) We use the terminologies adopted in Volterra's book: Theory of functionals and of integral and integro-differential equations, (1930). See specially Chapter IV, Theory of composition and of permutable and of permutable functions. We quote this by [V]. See also Volterra-Pérès, Leçons sur la composition et les fonctions permutables, (1924). We quote this by [$V.$-$P.$].

2) For its definition, see [V] Chapter IV, § 7, Order of a function, (p. 110), and [$V.$-$P.$] Chapter I, § 10 (p. 10).

3) See T. Kitagawa: (1) A Formulation of Operational Calculus, These Proceedings, **13** (1937). (2) A Theorem on Orerational Equation, These Proceedings, **13** (1937). We quote these by [F] and [E] respectively.

4) $\mathfrak{D}$ prossesses the certain special properties. See [F] § 2.

5) That is to say, $f(x, y)$ is defined in the domain $0 \leqq x \leqq y \leqq a$, and

$$f(x, x)=1, \qquad \left(\frac{\partial f(x, y)}{\partial x}\right)_{x=y} = \left(\frac{\partial f(x, y)}{\partial y}\right)_{x=y} = 0 \qquad (0 \leqq x \leqq a)$$

Further we assume that $\dfrac{\partial^2 f(x, y)}{\partial x \partial y}$ exists and is continuous with respect to (x, y). See [V] Chapter IV § 8, Group of the functions Permutable with a given Function (p. 111).

6) Each element $g(x, y)$ is defined and continuous in the domain $0 \leqq x \leqq y \leqq a$ and there

$$\int_x^y g(x, t) f(t, y)\, dt = \int_x^y f(x, t) g(t, y)\, dt,$$

that is to say,

$$\overset{**}{gf} = \overset{**}{fg}.$$

7) See [V] Chapter IV, § 9 Transformations which maintain the Law of Composition or Pérès's Transformations (p. 117). See also [$V.$-$P.$] Chapter IV, Les Transformations qui conservent la composition (pp. 56–67).