49. On Differential Operators permutable with Lie Continuous Groups of Transformations.

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1. In the present paper, we shall generalize Casimir's theorem[1] on semi-simple continuous groups, which may be stated as follows:

Let $X_1, X_2, \ldots\ldots, X_r$ generate a semi-simple continuous group and satisfy the law of compositions such that

$$[X_i, X_k] = C^{\alpha}_{ik} X_{\alpha}, \qquad (i, k = 1, 2, \ldots\ldots, r).$$

If (g^{ik}) denotes the inverse matrix of the coefficient matrix $(C^{\alpha}_{i\beta} C^{\beta}_{k\alpha})$ of Cartan's quadratic form

$$\varphi(\lambda, \lambda) = C^{\alpha}_{i\beta} C^{\beta}_{k\alpha} \lambda^i \lambda^k,$$

then the differential operator of the second order

$$P(X) = g^{ik} X_i X_k$$

is permutable with every element X_{ω}, that is,

$$X_{\omega} P(X) = P(X) X_{\omega}, \qquad (\omega = 1, 2, \ldots\ldots, r).$$

By means of this theorem, Profs. B. L. van der Waerden,[2] H. Casimir and Richard Brauer[3] gave the algebraic proof of Weyl's theorem[4] that all reducible representations of semi-simple continuous group are completely reducible.

2. In general, we assume that an r-parametric continuous group G of transformation is generated by r infinitesimal transformations

$$X_{\omega} = \xi^k_{\omega}(x^1, x^2, \ldots\ldots, x^n) \frac{\partial}{\partial x^k}, \qquad (\omega = 1, 2, \ldots\ldots, r),$$

where $\xi^k_{\omega}(x^3, x^2, \ldots\ldots, x^n)$ are analytic in a neighborhood of the origin. Then, we consider the symmetric differential operators of the ν-th order, defined as follows:

$$P_0(X) = g, \qquad P_1(X) = g^i X_i, \qquad P_2(X) = g^{ik} X_i X_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots, \qquad P_{\nu}(X) = g^{ikj\ldots l} X_i X_k X_j \ldots\ldots X_l,$$

where

$$g^{ik} = g^{ki}, \qquad \ldots\ldots\ldots, \qquad g^{ikj\ldots l} = g^{kij\ldots l},$$
$$g^{ikj\ldots l} = g^{ijk\ldots l} \quad \text{etc.},$$

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1) H. Casimir: Proc. Kon. Acad. Amsterdam, **34** (1931), 844.
K. Toyoda: Japanese Journal of Mathematics, **12** (1935), 17.
2) H. Casimir und B. L. van der Waerden: Math. Annalen, **111** (1935), 1.
3) R. Brauer: Math. Zeitschrift, **41** (1936), 330.
4) H. Weyl: Math. Zeitschrift, **24** (1926), 328.