10. *Notes on Fourier Series (III): Absolute Summability.*

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1. Let

$$\sum_{n=0}^{\infty} a_n \tag{1}$$

be a series such that

$$\sum_{n=0}^{\infty} a_n \rho^n \tag{2}$$

is convergent for positive $\rho < 1$. We denote (2) by $f(\rho)$. If $f(\rho)$ is of bounded variation in (0,1), that is

$$\int_0^r |f'(\rho)|\, d\rho \qquad (0 < r < 1)$$

is bounded, then we say that (1) is absolutely summable (A) or simply summable $|A|$.[1] The absolutely convergent series is summable $|A|$ and the series summable $|A|$ is summable (A).

Let $f(x)$ be an integrable function, periodic with period 2π, and its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)\,. \tag{3}$$

Let $\{\lambda_n\}$ be a sequence of real numbers. If the trigonometrical series

$$\sum_{n=0}^{\infty} \lambda_n A_n(x) \tag{4}$$

is summable $|A|$ for almost all x, then $\{\lambda_n\}$ is called the absolutely summable factor of (3).

B. N. Prasad[2] proved that if λ_n is one of the following[3]

$$\frac{1}{(\log n)^{1+\delta}}, \quad \frac{1}{\log n (\log_2 n)^{1+\delta}}, \quad \frac{1}{\log n \log_2 n (\log_3 n)^{1+\delta}}, \cdots\cdots \quad (\delta > 0) \tag{5}$$

then $\{\lambda_n\}$ is the absolutely summable factor. We will prove that if $\{\lambda_n\}$ tends to zero and is convex and further

$$\sum_{n=2}^{\infty} \log n \cdot \Delta\lambda_n \tag{6}$$

converges, then $\{\lambda_n\}$ is an absolutely summable factor. If λ_n tends to zero monotonously, then the convergence of (6) is equivalent to that of

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n}\,.$$

1) J. M. Whittaker, Proc. Edinburgh Math. Soc., **2** (1930–1931).
2) B. N. Prasad, Proc. London Math. Soc., **35** (1933).
3) $\log_2 n = \log(\log n)$, $\log_k n = \log(\log_{k-1} n)$ for $k > 2$.