75. *Iteration of Linear Operations in Complex Banach Spaces.*

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§ **1.** Introduction. In a recent paper C. Visser[1] discussed the iteration of linear operations in a Hilbert space and proved the following theorems:

Theorem I. Let E be a Hilbert space and A a linear operator which maps E in itself. If $\{A^n\}$ $(n=1, 2, \dots)$ is uniformly bounded, then there exists a bounded linear operator A_1, which maps E in itself, such that $\frac{1}{n}(A+A^2+\cdots+A^n)$ converges weakly to A_1.

Theorem II. If in addition to the assumptions in Theorem I, A is completely continuous, then the weak convergence in Theorem I may be substituted by the strong convergence.

In Visser's proof, the notion of inner product is indispensable, and therefore it is not applicable to the case of general Banach spaces. It is the purpose of the present paper to show that these theorems are also valid in general complex Banach spaces. Moreover, we can show that even in general complex Banach spaces the *weak* convergence in Theorem I may be substituted by the *strong* one (Theorem 2)[2] and that the *strong* convergence in Theorem II by the *uniform* one (Theorem 4). The former is a generalisation of J. v. Neumann's Mean Ergodic Theorem[3] and the latter is an analogue of M. Fréchet's theorem;[4] and following the same way, the results of N. Kryloff and N. Bogolioùboff[5] are also generalised to the case of general complex Banach spaces (Theorem 5).

The results of this paper are obtained in collaboration with K. Yosida. Theorem 2 is obtained by him directly,[6] and Theorem 4 and 5 are also obtained by him in different ways.[7] I shall, however, give the outline of my proof, since we can treat all these problems in a unique way and since the method itself, I believe, is not without interest. In concluding the introduction, I should like to express my hearty thanks to K. Yosida for his kindness in the course of this work.

1) C. Visser: On the iteration of linear operations in a Hilbert space, Proc. Acad. Amsterdam, **41** (1938), 487–495.

2) This is an essential advance! This is proved by K. Yosida. See the foregoing paper of K. Yosida: Mean Ergodic Theorem in Banach spaces.

3) J. v. Neumann: Proof of the quasi-ergodic hypothesis, Proc. Nat. Acad. U.S.A. **18** (1932), 70–82.

4) M. Fréchet: Sur l'allure asymptotique de la suite des itérés d'un noyau de Fredholm, Quart. Journ. of Math., **5** (1934), 106–144.

5) N. Kryloff et N. Bogolioùboff: Sur les probabilités en chaîne, C. R., **204** (1937), 1386–1388.

6) See the paper of K. Yosida cited in (2).

7) See the foregoing paper of K. Yosida: Abstract integral equation and the homogeneous stochastic process.