82. *A Generalization of Poincaré-space.*

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The set of symmetrical matrices A of dimension n satisfying the relation $E-A\bar{A}>0$ is called the space $\mathfrak{A}$ and A its points. $\mathfrak{A}$ is bounded, convex, and the points A satisfying the relation $|E-A\bar{A}|=0$ make the boundary of the space $\mathfrak{A}$.

Let $U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ be $2n$-dimensional matrices with the properties (1) $U'JU=J$, (2) $U'S\bar{U}=S$, where $J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, $S=\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$, then we call the transformations of the space $\mathfrak{A}$ into itself $W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$ the displacements of the space $\mathfrak{A}$.

We took as a line element the expression $ds=\sqrt{Sp\,dA(E-\bar{A}A)^{-1}d\bar{A}(E-A\bar{A})^{-1}}$ invariant under displacements and regarded $\mathfrak{A}$ as a Riemannian space.[1] However it seems to me more natural to introduce another metric which we will investigate here.

Let $W_i=(U_1Z_i+U_2)(U_3Z_i+U_4)^{-1}$, then by the property (1), we get

$$(W_1-W_4)^{-1}(W_1-W_3)(W_2-W_3)^{-1}(W_2-W_4)$$
$$=(U_3Z_4+U_4)(Z_1-Z_4)^{-1}(Z_1-Z_3)(Z_2-Z_3)^{-1}(Z_2-Z_4)(U_3Z+U_4)^{-1}.$$

Hence $h(Z_1, Z_2, Z_3, Z_4)=|Z_1-Z_4|^{-1}\,|Z_1-Z_3|\,|Z_2-Z_3|^{-1}\,|Z_2-Z_4|$ is invariant under displacements. We call it the "anharmonic ratio of the four ordered points Z_1, Z_2, Z_3, Z_4."

Especially $h(0, A, \lambda A, -\lambda A)=\left(\dfrac{1+\lambda^{-1}}{1-\lambda^{-1}}\right)^n$.

Let $\lambda_1 A$ and $\lambda_2 A$ be the intersecting points of the euclidean straight line $Z=\lambda A$, passing through 0 and $A\neq 0$, with the boundary of the space $\mathfrak{A}$, where λ varies over real numbers. Then $\lambda_2=-\lambda_1$ and λ_1 is the reciprocal of the positive quadratic root of the greatest proper value of the non-negative hermitian form $A\bar{A}$, $(A\neq 0)$[2]; for $|E-\lambda^2 A\bar{A}|=0$.

Now we define the distance $(0, A)$ between 0 and $A\in\mathfrak{A}$ as the quantity $\dfrac{1}{2n}\log h(0, A, \lambda_1 A, -\lambda_1 A)=\dfrac{1}{2}\log\dfrac{1+\lambda_1^{-1}}{1-\lambda_1^{-1}}$, then the distance $(0, A)>0$, because $0<\lambda_1^{-1}\leqq 1$; and $(0, A)\to 0$ when $A\to 0$.

We define $(0, 0)=0$. $(0, A)$ is invariant under any displacement fixing the point 0, $W=U_1AU_1'$, $U_1'\bar{U}_1=E$, because $W\bar{W}=U_1A\bar{A}\bar{U}_1'$ and $(0, A)$ depends only on a proper value of $A\bar{A}$.

Let B^* be the image of $B\in\mathfrak{A}$ by a displacement transforming $A\in\mathfrak{A}$ to 0. We define the distance (A, B) between two points A and

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1) Masao Sugawara, Über eine allgemeine Theorie der Fuchsschen Gruppen und Theta-Reihen. Ann. Math. **41**.

2) λ_1^{-1} is called the norm of a matrix A.