78. A Converse of Lebesgue's Density Theorem.

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I. The main object of the present note is to see that a converse of Lebesgue's density theorem holds.

We shall consider, for brevity, sets of points in a Euclidean plane R^2 only. But the results which will be obtained can obviously be extended to spaces R^m of any number of dimensions.

The Lebesgue outer measure of a set E in R^2 will be denoted by $|E|$. Let x be a point of R^2 and Q an arbitrary closed square containing x with sides parallel to the coordinate-axes.

We shall denote by $\overline{D}(x, E)$ and $\underline{D}(x, E)$ the superior and the inferior limit respectively of the ratio $|QE|/|Q|$ as the diameter of Q tends to 0 or $|Q|\to 0$, and shall call them the upper and the *lower density* of E at x respectively. If they are equal to each other at x, then the common value will be called the density of E at x. The points at which the density of E are equal to 0 are termed points of dispersion for E.

It is well known by Lebesgue's density theorem that, *if a set of points are measurable, almost every point of its complementary set is a point of dispersion for the given set.*[1]

II. We shall prove the following theorem which evidently contains a converse of the above proposition.

Theorem 1. *Let E be a point-set whose lower density is* 0 *at almost every point of the complementary set of E. Then the set E is measurable.*

Proof. We can obviously assume, without loss of generality, that the set E is bounded. Let G be a bounded open set containing E and ε a given positive number. From the assumption of the present theorem, there exists, for almost every point x of $H=G-E$, a sequence of squares $\{Q_n(x)\}$ such that

$$(1) \quad \frac{|E\cdot Q_n(x)|}{|Q_n(x)|}<\varepsilon,\ x\in Q_n(x),\ |Q_n(x)|\to 0\ (n\to\infty)\ \text{and}\ Q_n(x)\subset G.$$

Denoting by $\mathfrak{F}$ the family of all the squares which belong to any one of such sequences, we find that $\mathfrak{F}$ covers the set H almost everywhere in the sense of Vitali.[2] According to the covering theorem of Vitali[2] we can extract from $\mathfrak{F}$ a finite or enumerable sequence $\{Q_n\}$ of squares no two of which have common points, such that

$$|H-\sum_n Q_n|=0\,.$$

1) For example, S. Saks. Théorie de l'intégrale (1933), p. 55.
2) Saks. Loc. cit. 33–35.