101. Concircular Geometry III. Theory of Curves.

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In the two recent papers "Concircular Geometry I, and II[1]," we have considered concircular transformations $\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$ of a Riemannian metric $ds^2=g_{\mu\nu}du^\mu du^\nu$, that is to say, conformal transformations $\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$ with the function ρ satisfying

$$\rho_{\mu\nu}\equiv\frac{\partial\rho_\mu}{\partial u^\nu}-\rho_\lambda\left\{{}^{\lambda}_{\mu\nu}\right\}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu},$$

where ρ_μ denotes $\partial\log\rho/\partial u^\mu$ and $\left\{{}^{\lambda}_{\mu\nu}\right\}$ the three-index symbols of Christoffel formed with $g_{\mu\nu}$, and we have discussed the integrability conditions of these partial differential equations.

The purpose of the present note is to develop the theory of curves in the concircular geometry.

§ 1. *Frenet formulae.* Let us consider a curve $u^\lambda(s)$ in a Riemann space, s being the curve length measured from a fixed point on the curve, and form the vector

$$V^\lambda=\frac{\delta^3u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2u^\mu}{\delta s^2}\frac{\delta^2u^\nu}{\delta s^2} \tag{1.1}$$

where $\dfrac{\delta}{\delta s}$ denotes the covariant differentiation along the curve.

If we effect a conformal transformation of the metric

$$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}, \tag{1.2}$$

the vector V^λ will be transformed into

$$\bar{V}^\lambda=\frac{1}{\rho^3}\left[V^\lambda+\frac{\delta u^\lambda}{\delta s}\rho_{\mu\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}-g^{\lambda\alpha}\rho_{\alpha\nu}\frac{\delta u^\nu}{\delta s}\right]. \tag{1.3}$$

Hence, if the conformal transformation (1.2) is a concircular one, that is to say, if the function ρ satisfies

$$\rho_{\mu\nu}=\phi g_{\mu\nu}, \tag{1.4}$$

the equations (1.3) become

$$\bar{V}^\lambda=\frac{1}{\rho^3}V^\lambda, \tag{1.5}$$

which shows that the direction defined by the vector V^λ is invariant under a concircular transformation.

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1) K. Yano: Concircular Geometry I, Proc. **16** (1940), 195–200, and Concircular Geometry II, Proc. **16** (1940), 354–360.