# 119. Normal Basis of a Quasi-field.

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Recently N. Jacobson extended the fundamental theorem of the Galois theory to *quasi-fields* in the following sense[1]: Let $P$ be a quasi-field and there be given a *finite group of outer automorphisms*[2] $\mathfrak{G}=\{E, S, \dots, T\}$, of order, say $n$. If $\Phi$ is the sub-quasifield of invariant elements, then $P$ has the rank $n$ over $\Phi$ (at both left and right) and there exists a 1-1 correspondence between subgroups of $\mathfrak{G}$ and sub-quasifields between $P$ and $\Phi$. The purpose of the present note is to show that moreover $P$ possesses a (one-sided) normal basis[3] over $\Phi$, that is, there exists an element $b$ in $P$ such that the $n$ conjugates, so to speak, $b^E, b^S, \dots, b^T$ of $b$ form a (linearly independent) left (say)-basis of $P$ over $\Phi$. The proof is a generalization of M. Deuring's *second* proof to the theorem of commutative normal bases;[4] the proof has been emancipated, by the present writer,[5] from the restriction on the semisimplicity of the group ring. But it involves modifications caused by the non-commutativity and makes use of a generalization of the Hilbert-Speiser theorem in a *refined* form.

Let $P$, $\mathfrak{G}$, $n$ and $\Phi$ be as above. Denote the center[6] of $P$ by $Z$, and put $K=\Phi \frown Z$. Let further $K^*$ be a finite extension of $K$, and let

$$P^*=P_{K^*}, \qquad \Phi^*=\Phi_{K^*}$$

be the rings obtained from $P$ and $\Phi$ by extending the ground field $K$ to $K^*$. (They are not, in general, quasi-fields any more). Automorphisms $E, S, \dots, T$ of $P$ can be looked upon, in natural manner, as those of $P^*$ (and in fact $\Phi^*$ consists of the totality of invariant elements).

*Lemma 1* (Generalized Hilbert-Speiser theorem). *Let to each $S$ in*

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1) N. Jacobson, The fundamental theorem of Galois theory for quasi-fields, Ann. Math. **41** (1940).

2) We mean that all the automorphisms in $\mathfrak{G}$ except the identity are outer.

3) For the theorem of normal basis of a commutative field see: E. Noether, Normalbasis bei Körpern ohne höhere Verzweigung, Crelle, **167** (1931); M. Deuring, Galoissche Theorie und Darstellungstheorie, Math. Ann. **107** (1932); H. Hasse, Klassenkörpertheorie, Marburg (1932); R. Brauer, Über die Kleinsche Theorie der algebraischen Gleichungen, Matn. Ann. **110** (1934); M. Deuring, Anwendungen der Darstellungen von Gruppen durch linearen Substitutionen auf die Galoissche Theorie, Math. Ann. **113** (1936); R. Stauffer, The construction of a normal basis in a separable normal extension field, American J. Math, **58** (1936). There is also an unpublished proof by E. Artin.

4) M. Deuring, Math. Ann. **110**, l.c.

5) T. Nakayama, On Frobeniusean algebras, II (forthcoming in Math. Ann.), § 3. Appendix.

6) We are interested only in the case where $P$ has an *infinite rank* over its center. For, otherwise the theorem can readily be reduced to the commutative case, because of Jacobson's result.