116. *An Abstract Integral, III.*

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The object of this paper is to make the integration theory free from the concept of function.

1. Let $\mathbb{L}$ be a system of elements $a, b, c, \dots, x, y, z, \dots$ and let $\alpha, \beta, \gamma, \dots$ be real numbers and $k, m, n, \dots$ be integers. We suppose that $\mathbb{L}$ satisfies the following axioms.

Axiom 1. $\mathbb{L}$ is an abelian group with real number field as operator domain. Group operation is denoted by " $+$ ".

Axiom 2. $\mathbb{L}$ is partially ordered, that is, the relation " $\leqq$ " is defined and

(2.1) $a \leqq a$,

(2.2) $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

Axiom 3. $\mathbb{L}$ is a lattice, that is, for every a and every b in $\mathbb{L}$, there exist the join $a \cup b$ and the meet $a \cap b$ such that

(3.1) $a \leqq a \cup b$, $b \leqq a \cup b$, and $a \leqq c$, $b \leqq c$ imply $a \cup b \leqq c$,

(3.2) $a \geqq a \cap b$, $b \geqq a \cap b$, and $a \geqq d$, $b \geqq d$ imply $a \cap b \geqq d$.

Axiom 3′. $\mathbb{L}$ is a "restricted" σ-lattice, that is, for any "bounded"[1] sequence $\{x_n\}$, there exist the elements $\bigvee_{n=1}^{\infty} x_n$ and $\bigwedge_{n=1}^{\infty} x_n$ such that

(3′.1) $x_m \leqq \bigvee_{n=1}^{\infty} x_n$ $(m=1, 2, \dots)$ and $x_n \leqq c'$ $(n=1, 2, \dots)$ imply $\bigvee_{n=1}^{\infty} x_n \leqq c'$,

(3′.2) $x_m \geqq \bigwedge_{n=1}^{\infty} x_n$ $(m=1, 2, \dots)$ and $x_n \geqq d'$ $(n=1, 2, \dots)$ imply $\bigwedge_{n=1}^{\infty} x_n \geqq d'$.

Axiom 4. Between partially ordering and group operation there hold the relations:

(4.1) $a > 0$ implies $-a < 0$,

(4.2) $a > b$ implies $a+c > b+c$,

(4.3) $\alpha > 0$ and $a > 0$ imply $\alpha a > 0$.

We need further some definitions.

Definition 1. $x^+ = x \cup 0$, $x^- = x \cap 0$ and $|x| = x^+ - x^-$.

Definition 2. $\overline{\lim\limits_{n\to\infty}}\, x_n = \bigwedge_{n=1}^{\infty} (\bigvee_{m=n}^{\infty} x_m)$, $\underline{\lim\limits_{n\to\infty}}\, x_n = \bigvee_{n=1}^{\infty} (\bigwedge_{m=n}^{\infty} x_m)$, provided that $\{x_n\}$ is bounded. If they coincide, then we denote it by $\lim\limits_{n\to\infty} x_n$.

2. We will now define the abstract Riemann and Lebesgue integral of element of $\mathbb{L}$. We will begin by the

1) Let $S < \mathbb{L}$. If there are u and l in $\mathbb{L}$ such that $l \leqq s \leqq u$ for all s in S, then S is called bounded.