103. On the General Schwarzian Lemma.

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We aim to generalize the Schwarzian lemma in the theory of functions of one variable to the case of the higher dimension and apply it to the characterization of the displacements of the general Poincaré-space.

Definition 1. Let $W^{(m,n)}=(w_{ij})$ and $Z^{(m,n)}=(z_{ij})$ be two matrices of (m,n) type. We call W *an analytic function* $f(Z)$ of Z, if the elements w_{ij} of W are analytic functions of the elements z_{ij} of Z, it is called *regular at a point* Z, if w_{ij} are regular functions of the elements z_{ij} in a neighbourhood of Z, and it is called *regular in a domain* D of Z, if it is regular at every point of D.

Definition 2. Let $A^{(m,n)}$ be a matrix and $\mathfrak{x}$ be a n-dimensional vector with the length 1. Put $|A|=\underset{|\mathfrak{x}|=1}{\text{l. u. b.}}|A\mathfrak{x}|$.

We call $|A|$ *the absolute value* of A or *the norm* of A.

Definition 3. A function W of Z is called *partially constant,* if there exist two constant unitary matrices U, V of the dimension m and n resp. such that $UWV=(a_{ij})$ in which a_{ii}, $i=1,2,\dots,r$ are all constant and all the elements $a_{ij}=0$, where $i\neq j$ and $i\leqq r$ or $j\leqq r$.

Theorem 1. When $f(Z)$ is regular in a closed domain D of Z, the maximum absolute value of $f(Z)$ is taken on the boundary of D. It is taken also at inner points of D, when and only when W is partially constant.

Definition 4. A matrix is said to be of a *D-form,* if it is of the form $(a_{ij}\delta_{ij})$ in which δ_{ij} means the Kronecker's symbol; namely $\delta_{ij}=1$, if $i=j$, and $=0$ in other cases.

Definition 5. A matrix B of a D-form is said to be a *N-form* of a matrix A, if there exist two constant unitary matrices U and V such that $A=UBV$.

Proof of the theorem 1. Let $Z_0=(z_{ij}^0)$ be an inner point of D at which $f(Z)$ takes its maximum absolute value. We can evidently assume that $f(Z_0)\neq 0$. Take two constant unitary matrices U and V such that $Uf(Z_0)V$ is a D-form $(z_i\delta_{ij})$ and $|f(Z_0)|=|z_1|$.

Put $f_1(Z)=Uf(Z)V=(x_{ij})$, then $f_1(Z)$ is also a regular analytic function of Z in D such that $|f_1(Z)|=|f(Z)|$ and $|x_{11}^0|=|z_1|$, x_{11}^0 being the value of x_{11} at the point Z_0. As Z_0 is an inner point of D we can find a domain $F=(Z;|z_{ij}-z_{ij}^0|\leqq\varepsilon)$ in D, if we take ε sufficiently small. As x_{11} takes its maximum absolute value on the boundary of F and as $|f(Z_0)|=|x_{11}^0|\geqq|f_1(Z)|\geqq|x_{11}|$, x_{11} is a constant. Hence $x_{i1}=0$ $(i=2,\dots,m)$, as $|f(Z)|^2\geqq\sum_{k=1}|x_{k1}|^2$ and $x_{1j}=0$ $(j=2,\dots,n)$, because $|f(Z)|^4\geqq|x_{11}|^2(|x_{11}|^2+|x_{12}|^2+\cdots+|x_{1n}|^2)$; the right-hand side of the last inequality being the square of the length of the 1st column-