PAPERS COMMUNICATED

1. Note on Lattice-ordered Groups.

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By a lattice-ordered group, or briefly a lattice-group, we mean a (not necessarily abelian) group which is at the same time a lattice such that the order relation is preserved under left and right multiplication; $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$; we have $ac \asymp bc = (a \asymp b)c$, $ca \asymp cb = c(a \asymp b)$ too.

Abelian lattice-groups[1)], particularly so-called vector lattices[2)], have been studied by many authors. The present short note[3)] is to give some simple remarks concerning mainly with non-abelian lattice-groups. We shall begin with elementary observations about homomorphisms. We shall then show that Lorenzen's main theorem for abelian lattice-groups can be transfered to the non-abelian case with minor modifications. However, this does not give, contrary to Clifford's abelian case, a representation of the lattice-group by linearly ordered ones: it gives merely a representation of the lattice-group by linearly ordered systems of cosets with respect to its subgroups. It follows readily that every lattice-group is, considered as a lattice, distributive. This fact, however, can easily be seen also by modifying somewhat the well-known proofs to the distributivity of abelian lattice-groups[4)]. The structure of lattice-groups satisfying the conditional (= weak) maximum condition is very simple and rather trivial; they are necessarily abelian[5)6)]. We shall also observe that a recent result by Yosida-Fukamiya[7)] concerning

1) R. Dedekind, Über Zerlegung von Zahlen durch ihre grössten gemeinschaftlichen Teiler (Ges. Werke, Bd. 2, XXVIII); P. Lorenzen, Abstrakte Begründung der multiplikativen Idealtheorie, Math. Zeitschr. **45** (1939); A. H. Clifford, Partially ordered abelian groups, Ann. Math. **41** (1940).

2) L. V. Kantorovitch, Lineare halbgeordnete Räume, Mat. Shornik **2** (1937); H. Freudenthal, Teilweise geordnete Moduln, Proc. Amsterdam **39** (1936). For some of more recent literatures see the references in G. Birkhoff, Lattice theory, New York (1940); K. Yosida, On vector lattice with a unit, Proc. **17** (1941).

3) I want to express my thanks to Mr. K. Yosida for the useful remarks he gave me during the preparation of the present note.

4) For abelian case see Dedekind, l. c., Freudenthal, l. c. and Birkhoff, l. c.

5) As a matter of fact, the essential feature of this fact is contained already in the commutativity of two-sided ideals in the arithmetical theory of algebras, non-commutative polynomicals and non-commutative semi-groups. Besides early works by E. Artin, O. Ore and others, cf. K. Asano, Arithmetische Idealtheorie in nichtkommutativen Ringen, Jap. Journ. Math. **16** (1939); Y. Kawada-K. Kondo, Idealtheorie in nicht-kommutativen Halbgruppen, ibid. **16** (1939); T. Nakayama, A note on the elementary divisor theory in non-commutative domains, Bull. Amer. Math. Soc. **44** (1938).

6) This is a very simple; and trivial, special case of G. Birkhoff's conjecture that conditionally complete lattice-groups will always be abelian. The conjecture was communicated to me by S. Kakutani.

7) K. Yosida-M. Fukamiya, On vector lattice with a unit, II., Proc. **17** (1941).