PAPERS COMMUNICATED

116. *On Locally Convex Topological Spaces.*

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Let L be a vector space and D a directed system. If there exists a real valued function $|x|_d$ on the domain $L \times D$ such that

(1) $|x|_d \geqq 0$; if $|x|_d=0$ for all $d \varepsilon D$ then $x=\theta$,

(2) $|ax|_d=|a|\cdot|x|_d$ for any real a,

(3) for any given $e \varepsilon D$ there exists $d \varepsilon D$ such that $|x|_d \to 0$ and $|y|_d \to 0$ imply $|x+y|_e \to 0$,

(4) $d < e$ implies $|x|_d \leqq |x|_e$,

then L is said to be a pseudo-normed linear space. It is proved by D. H. Hyers [1] that the pseudo-normed linear space is a linear topological space, which was defined by A. Kolmogoroff [2] and J. v. Neumann [3]. The triangular inequality

(3′) $|x+y|_d \leqq |x|_d+|y|_d$

is stronger than (3). If we take therefore the condition (3′) instead of (3) in addition of (1), (2), (4), then the space L is said to be a locally convex linear topological space. In this paper we concern the locally convex linear topological space L and its conjugate spaces $\bar{L}$ and $\bar{\bar{L}}$.

§ **1.** *Space* $\bar{L}$. The family of the sets $u(d, \delta) \equiv (x; |x|_d < \delta)$ $(\delta > 0)$ is said to be a fundamental system of the origin θ; we denote it by $\{u(d, \delta); \delta > 0\}$.

Theorem 1. Referring the fundamental system $\{u(d, \delta); \delta>0\}$, L is a locally convex linear topological space.

For a linear functional $f(x)$ on the domain L, if there exist some $d \varepsilon D$ and $M(d) > 0$ such that

(1) $|f(x)| \leqq M(d)\cdot|x|_d$ for all $x \varepsilon L$,

then $f(x)$ is said to be bounded.

Theorem 2. For linear functionals continuity is equivalent to boundedness.

For the linear continuous functional $f(x)$ the set of all d with condition (1) is denoted by D_f, and for a given $d \varepsilon D$ the set of all $f(x)$ with condition (1) is denoted by $\bar{L}_d$.

Theorem 3. D_f is a cofinal subsystem of D.

Proof. If d' and d'' are two elements of D_f, then $|f(x)| \leqq M(d')\cdot |x|_{d'}$, and $|f(x)| \leqq M(d'')\cdot|x|_{d''}$ for all $x \varepsilon L$. Since D is a directed system, there exists a d such that $d'<d$ and $d''<d$. Consequently $|f(x)| \leqq M(d')\cdot|x|_d$ and $|f(x)| \leqq M(d'')\,|x|_d$ for all $x \varepsilon L$. That is $d \varepsilon D_f$. For any d in D there exists d'' such as $d''>d$ and $d''>d'$, so that $|f(x)| \leqq M(d')\,|x|_{d'} \leqq M(d')\,|x|_{d''}$, which shows that D_f is a cofinal subsystem of D.

Theorem 4. (i) $|f|_d \geqq 0$; and if $|f|_d=0$ then $f(x) \equiv 0$,