24. *Some Metrical Theorems on a Set of Points.*

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In this note we will prove some theorems on measurable sets of points.

Theorem I. Let E be a measurable set in an n-dimensional space. We translate E by a vector $\mathfrak{r}$ and $E+\mathfrak{r}$ be the translated set. Then

$$\lim_{|\mathfrak{r}|\to 0} mE(E+\mathfrak{r})=mE. \tag{1}$$

W. H. Young[1] proved the case $n=1$.

Proof. We prove the case $n=2$; the other case can be proved similarly. Let E be a measurable set on the xy-plane and $\varphi(x, y)$ be its characteristic function, then $\varphi(x-h, y-k)$ is the characteristic function of $E+\mathfrak{r}$, where (h, k) are the components of $\mathfrak{r}$, so that $\mathfrak{r}=(h, k)$, $|\mathfrak{r}|=\sqrt{h^2+k^2}$.

(i) First we assume $mE<\infty$. Then

$$mE=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x, y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^2(x, y)dxdy,$$

$$mE(E+\mathfrak{r})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x, y)\varphi(x-h, y-k)dxdy,$$

so that

$$|mE(E+\mathfrak{r})-mE|=\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x, y)\big(\varphi(x-h, y-h)-\varphi(x, y)\big)dxdy\right|\leqq$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x-h, y-k)-\varphi(x, y)|\,dxdy.$$

Since by Lebesgue's theorem[2],

$$\lim_{h^2+k^2\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x-h, y-k)-\varphi(x, y)|\,dxdy=0,$$

we have $\lim\limits_{|\mathfrak{r}|\to 0} mE(E+\mathfrak{r})=mE$.

(ii) If $mE=\infty$, let E_1 be a bounded sub-set of E, such that $N\leqq mE_1<\infty$. Then by (i), for any $\mathfrak{r}$, such that $|\mathfrak{r}|<\rho$ $mE_1(E_1+\mathfrak{r})\geqq \frac{mE_1}{2}\geqq\frac{N}{2}$, so that $mE(E+\mathfrak{r})\geqq mE_1(E_1+\mathfrak{r})\geqq\frac{N}{2}$. Since N can be taken arbitrarily large, we have $\lim\limits_{|\mathfrak{r}|\to 0} E(E+\mathfrak{r})=\infty$, q. e. d.

Theorem II. Let E_1 and E_2 be measurable sets in an n-dimensional space and one of mE_1, mE_2 be finite. Then

$$\lim_{|\mathfrak{r}|\to 0} mE_1(E_2+\mathfrak{r})=m(E_1\cdot E_2). \tag{2}$$

1) W. H. Young: On a class of parametric integrals and their application in the theory of Fourier series. Proc. Royal Soc. (London) A. **85** (1911).

2) Lebesgue: Lecons sur les séries trigonométriques. p. 15.