109. *Bohr Compactifications of a Locally Compact Abelian Group II.*

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This is a continuation of our preceding paper with the same title.[1] As an application of the general theory developed in the first paper, we shall here discuss monothetic and solenoidal groups.

§ 5. *Monothetic groups.* A topological group G is *monothetic*,[2] if there exists an element $a \in G$, called a *generating element* of G, such that the cyclic subgroup $H=\{a^n \mid n=0, \pm 1, \pm 2, \dots\}$ of H generated by a is everywhere dense in G. A monothetic group is obviously abelian.

Theorem 5.[3] *A locally compact monothetic group G is either compact or topologically isomorphic with the additive group of all integers with discrete topology.*

Proof. Let us apply Theorem 1, taking as H the additive group of all integers with discrete topology. Then there exists a continuous isomorphism $a^{*\prime}=\varphi^*(a^*)$ of the character group G^* of G onto a topological subgroup $G^{*\prime}$ of the character group H^* of H which is nothing but the additive group K of all real numbers mod. 1 with the usual compact topology. It suffices to show that G^* is either compact or discrete. If G^* is not totally disconnected, then the image $\varphi^*(V^*)$ in K of every open neighborhood V^* of the zero element of G^* contains a continuum and hence a certain interval of K containing the zero element of K. From this follows that $a^{*\prime}=\varphi^*(a^*)$ is an open mapping and so is a homeomorphism. Thus, if G^* is not totally disconnected, then G^* must be topologically isomorphic with K. This, however, happens, only if G is discrete and is topológically isomorphic with H itself. On the other hand, if G^* is totally disconnected, then there exists an arbitrary small subgroup of G^* which is open-and-closed. But this is possible only if G^* is discrete; for, as a topological subgroup of K, the continuous image $G^{*\prime}$ of G^* has no sufficiently small subgroup except the trivial one consisting only of the zero element. Thus G^* must be discrete in this case, and so $G=G^{**}$ must be compact. This completes the proof of Theorem 5.

Remark 5. It is not difficult to construct an example of a complete metric monothetic group which is not locally compact. In fact, let $f^*(n)$ be a complex-valued bounded function defined on the additive group $H=\{n \mid n=0, \pm 1, \pm 2, \dots\}$ of all integers with the following properties: (i) there exists a sequence $\{m_k \mid k=1, 2, \dots\}$ of positive

1) H. Anzai and S. Kakutani, Proc. **19** (1943), 476–480.

2) Cf. D. van Dantzig, Compositio Math. **3** (1936), 408–426.

3) This theorem is due to A. Weil, L'intégration dans les groupes et leurs applications, Actualités, 1939. The proof given in this paper is new.