PAPERS COMMUNICATED

15. Completely Continuous Transformations in Hilbert Spaces.

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1. By a space of type A[1] we mean a Banach space in which there exist a linearly independent sequence $\{f_n\}$ of elements of unit norm and a double sequence $\{L_{mn}(f)\}$ of bounded linear functionals such that for every f

$$\text{(A)} \qquad \lim_{m\to\infty} \| f - \sum_{n=1}^{m_n} L_{mn}(f) f_n \| = 0 .$$

It will be seen that the conception of a space of type A is a generalization of the idea of a Banach space with a denumerable base[2].

Let $\mathfrak{T}$ denote the space of all completely continuous transformations of a Hilbert space $\mathfrak{H}$ into itself, that is, the space of all bounded linear transformations which carry every bounded set of $\mathfrak{H}$ into a compact set.

In this note we will prove that the space $\mathfrak{T}$ is a separable space of type A.

2. We prove now the following theorem:

Theorem 1. In the space $\mathfrak{T}$, there exist a linearly independent double sequence $\{T_{ij}\}$ of elements of unit norm and a double sequence $\{a_{ij}(T)\}$ of bounded linear functionals such that for any $T \in \mathfrak{T}$

$$T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}(T) T_{ij} .$$

Proof. Let $\{\varphi_n\}$ denote the complete orthonormal set of the space $\mathfrak{H}$. We define $\{T_{ij}\}$ as follows;

$$T_{ij}(x) = (x, \varphi_j)\varphi_i \quad \text{for all} \quad x \in \mathfrak{H} , \qquad (i, j = 1, 2, \dots) .$$

Then it is evident that $T_{ij} \in \mathfrak{T}$, $\| T_{ij} \| = 1$ and the sequence $\{T_{ij}\}$ is linearly independent. Let $\mathfrak{M}_j$ be the closed linear manifold determined by $\{\varphi_1, \varphi_2, \cdots, \varphi_j\}$. Then we can prove that every bounded linear transformation T with domain $\mathfrak{H}$ and with range $\mathfrak{M}_1$ is expressed in the form $T = \sum_{j=1}^{\infty} a_{1j}(T) T_{1j}$ where $a_{1j}(T)$ are bounded linear functionals.

In fact, by use of F. Riesz' theorem[3] it can be easily shown that

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1) The notion of a space of type A was introduced by I. Maddaus. I. Maddaus; Completely continuous linear transformations, Bull. Amer. Math. Soc. Vol. 44 (1938), 279-282.

2) S. Banach; Théories des opérations linéaires, p. 110.

3) M. H. Stone; Linear transformations in Hilbert space and their applications to analysis, p. 62, Theorem 2. 27.