"Shafarevich-Tate sets" for profinite groups

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1. Ш(*G*) for a topological group *G*. Let G be a topological group. By a cocycle of G we mean a continuous map $f : G \to G$ such that

$$(1.1)\qquad f(st) = f(s)f(t)^s, \text{ with } a^s = sas^{-1}, \quad s, t \in G, \ a \in G.$$

We denote by $Z(G)$ the set of all cocycles. Two cocycles f, f' are *equivalent*, written $f \sim f'$, if there is an $a \in G$ such that

$$(1.2)\qquad f(s) = a^{-1}f'(s)a^s, \ s \in G.$$

Cocycles f, f' are *locally equivalent*, written $f \underset{\text{loc}}{\sim} f'$, if there is an $a_s \in G$ for each $s \in G$ such that

$$(1.3)\qquad f(s) = {a_s}^{-1}f'(s){a_s}^s.$$

Two subsets $B(G)$, $B_{\text{loc}}(G)$ of $Z(G)$ are defined by

$$(1.4)\qquad B(G) = \{f \in Z(G); f \sim 1\}, \quad B_{\text{loc}}(G) = \{f \in (Z(G)); f \underset{\text{loc}}{\sim} 1\},$$

respectively, where 1 denotes the constant function on G of value 1_G. A cocycle in $B(G)$ is a coboundary and one in $B_{\text{loc}}(G)$ is a local coboundary. Clearly a coboundary is a local cobundary : $B(G) \subset B_{\text{loc}}(G)$. The S–T set Ш$(G)$ is the quotient of $B_{\text{loc}}(G)$ with respect to the equivalence (1.2). $B(G)$ forms a distinguished point in Ш(G). When Ш$(G) = 1$, i.e. $B_{\text{loc}}(G) = B(G)$, we say that G enjoys the "Hasse principle".

2. *Z*(*G*) and End(*G*). Let G be a topological group and End(G) the semigroup of continuous homomorphisms of G into G. I owe M. Mazur an excellent idea of associating an $F \in \text{End}(G)$ to each $f \in Z(G)$ by

$$(2.1)\qquad F(s) = f(s)s, \ s \in G.$$

It is easy to verify that the map $f \mapsto F$ is a bijection

$$(2.2)\qquad \mu : Z(G) \xrightarrow{\sim} \text{End}(G),$$

and $Z(G)$ becomes a semigroup with the multiplication

$$(2.3)\qquad f * f'(s) = f(f'(s)s)f'(s), \quad f, f' \in Z(G).$$

The equivalence in End(G) corresponding to the one in (1.2) turns out to be

$$(2.4)\quad F \sim F' \iff F(s) = a^{-1}F'(s)a, \ a \in G.$$

In particular, to a coboundary $f(s) = a^{-1}a^s$ corresponds the inner automorphism $F(s) = a^{-1}sa$. In other words, the map (2.2) induces a bijection

$$(2.5)\qquad B(G) \xrightarrow{\sim} \text{Inn}(G)$$

and $B(G)$ becomes a group.

Similarly, another equivalence in End(G) corresponding to the local equivalence (1.3) turns out to be

$$(2.6)\quad F \underset{\text{loc}}{\sim} F' \iff F(s) = {a_s}^{-1}F'(s)a_s \iff F(s) \sim F'(s),$$

the (pointwise) conjugacy in G.

In particular, to a local coboundary $f(s) = {a_s}^{-1}{a_s}^s$ corresponds an endomorphism F such that $F(s) \sim s$. In other words, the map (2.2) induces a bijection

$$(2.7)\qquad B_{\text{loc}}(G) \xrightarrow{\sim} \text{End}_c(G)$$

where the right hand side is the set of $F's$ such that $F(s) \sim s$, i.e., the set of endomorphisms which preserve conjugacy classes of G. It should be noted that every F in End$_c(G)$ is injective but not surjective in general. Denoting by i_a the inner automorphism of G such that $i_a(s) = asa^{-1}$, we have, for $F, F' \in \text{End}_c(G)$,

$$(2.8)\qquad F \sim F' \iff F'(s) = aF(s)a^{-1} \iff F' = i_aF, \ a \in G.$$

Consequently, we obtain a bijection

$$(2.9)\quad Ш(G) \approx B(G) \setminus B_{\text{loc}}(G) \approx \text{Inn}(G) \setminus \text{End}_c(G).$$

Let Aut(G) be the group of automorphisms of G. Set

$$(2.10)\qquad \text{Aut}_c(G) = \text{Aut}(G) \cap \text{End}_c(G)$$

the subgroup of Aut(G) preserving conjugacy classes of G. Therefore if the condition

(♯) *every F in* End$_c(G)$ *is surjective and F^{-1} is continuous*