Spectra of categories

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1. Introduction. We define the "Laplacian" or the "adjacency matrix" of a category $\mathcal{C}$ via

$$\Delta(\mathcal{C}) = \left({}^{\#}\mathrm{Hom}_{\mathcal{C}}(X,Y)\right)_{X,Y\in \mathrm{Ob}(\mathcal{C})}$$

where $\mathrm{Ob}(\mathcal{C})$ is the "set" (or "class") of objects, and $^{\#}$ denotes the cardinality. This notion is borrowed from the graph theory (cf. Biggs [1]), since a category is a certain "oriented graph" satisfying the associative law for edges (morphisms).

We are especially interested in the most basic case where $\mathcal{C}$ is consisting of abelian groups or modules. For convenience, when we are treating the category $\mathcal{C}$ consisting of finite abelian groups $A_1,\dots,A_n$, we denote the Laplacian $\Delta(\mathcal{C})$ concretely as

$$\Delta(A_1,\dots,A_n) = \left({}^{\#}\mathrm{Hom}(A_i,A_j)\right)$$

where $i,j=1,\dots,n$. More generally, for (left) R-modules $M_1,\dots,M_n$ over a ring R, we simply write the associated Laplacian as

$$\Delta_R(M_1,\dots,M_n) = \left({}^{\#}\mathrm{Hom}_R(M_i,M_j)\right).$$

Naturally $\Delta(A_1,\dots,A_n)=\Delta_{\mathbf{Z}}(A_1,\dots,A_n)$.

We hope to study the spectra (eigenvalues) $\mathrm{Spect}\Delta(\mathcal{C})$ of $\Delta(\mathcal{C})$. In general we expect that $\Delta(\mathcal{C})$ behaves like the classical Laplacian appearing in the differential geometry. In particular, $\Delta(\mathcal{C})$ would be symmetric and semi-positive, and the spectra would be distributed as usual.

Here we restrict ourselves to the case of $\Delta(A_1,\dots,A_n)$ and $\Delta_R(M_1,\dots,M_n)$ as well as their behavior as $n\to\infty$. Main results are as follows. First:

Theorem 1. *For finite abelian groups $A_1,\dots,A_n$, $\Delta(A_1,\dots,A_n)$ is a symmetric matrix.*

We conjecture that $\Delta(A_1,\dots,A_n)$ is semi-positive. (The case $n=2$ is proved in [3].) The next result gives an affirmative answer for $\Delta(\mathbf{F}_p^{m_1},\dots,\mathbf{F}_p^{m_n})$ where p is a prime.

Theorem 2. *Let $\mathbf{F}_q$ be a finite field of q elements. Then*

$$\Delta_{\mathbf{F}_q}(\mathbf{F}_q^{m_1},\dots,\mathbf{F}_q^{m_n}) = (q^{m_i m_j})$$

is a semi-positive matrix for integers $m_i\ge 0$.

Finally we examine the behavior of spectra as $n\to\infty$ in a simple situation.

Theorem 3. *Let p_n be the n-th prime. Then the spectra $\lambda_1^{(n)}\le\lambda_2^{(n)}\le\cdots\le\lambda_n^{(n)}$ of $\Delta(\mathbf{Z}/p_1\mathbf{Z},\dots,\mathbf{Z}/p_n\mathbf{Z})$ are all simple and located as*

$$p_1-1<\lambda_1^{(n)}<p_2-1<\lambda_2^{(n)}<\cdots<p_n-1<\lambda_n^{(n)}.$$

In particular, $\Delta(\mathbf{Z}/p_1\mathbf{Z},\dots,\mathbf{Z}/p_n\mathbf{Z})$ is a positive matrix. Moreover, for each fixed $m\ge 1$, we have

$$\lim_{n\to\infty}\lambda_m^{(n)} = p_m-1.$$

We remark that the convergence is very slow. For example $\lim_{n\to\infty}\lambda_1^{(n)}=1$, but $\lambda_1^{(100)}=1.25467\cdots$, $\lambda_1^{(1600)}=1.23294\cdots$, and roughly

$$\lambda_1^{(n)}\approx 1+\frac{1}{\log\log n}$$

as analyzed later.

It is well-known that spectra of Laplacians explain zeros and poles of zeta functions for Riemannian manifolds and graphs. Relations to categorical zeta functions in the direction of [2] will be treated at another occasion.

2. Symmetry. We prove Theorem 1. It is sufficient to prove the following

Lemma 1. *Let A and B be finite abelian groups, then*

$$^{\#}\mathrm{Hom}(A,B) = {}^{\#}\mathrm{Hom}(B,A).$$

Proof. Let $\hat{A}=\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})$, $\hat{B}=\mathrm{Hom}(B,\mathbf{Q}/\mathbf{Z})$ be the dual abelian groups. (We describe abelian groups additively.) There is a natural homomorphism

$$\begin{array}{cccc}\varphi: & \mathrm{Hom}(A,B) & \longrightarrow & \mathrm{Hom}(\hat{B},\hat{A})\\ & \Psi & & \Psi\\ & f & \longmapsto & \varphi(f)\end{array}$$

defined via

$$\varphi(f)(\chi)=\chi\circ f \qquad \text{for } \chi\in\hat{B}.$$