Bernstein degree of singular unitary highest weight representations of the metaplectic group

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Let ω be the Weil representation of the metaplectic double cover $G = Mp(2n, \mathbf{R})$ of the symplectic group $Sp(2n, \mathbf{R})$ of rank n. Consider the m-fold tensor product $\omega^{\otimes m}$ of ω. Then the orthogonal group $O(m)$ acts on $\omega^{\otimes m}$ from the right and the action generates the full algebra of intertwiners. Therefore we can decompose $\omega^{\otimes m}$ as $G \times O(m)$-module (see [6, 7]):

$$\omega^{\otimes m} = \bigoplus_{\sigma \in \widehat{O}(m)} L(\sigma) \otimes \sigma.$$

In this article, we consider $L(\mathbf{1}_m)$ $(1 \le m \le n)$ which corresponds to the trivial representation $\mathbf{1}_m$ of $O(m)$. If $1 \le m \le n$, $L(\mathbf{1}_m)$ is an irreducible singular unitary highest weight representation of G and it has one-dimensional minimal K-type. Note that, if m is even, then $L(\mathbf{1}_m)$ factors through and gives an irreducible representation of $Sp(2n, \mathbf{R})$.

The aim of this article is to give a formula for the Bernstein degree of $L(\mathbf{1}_m)$, which is denoted by $\operatorname{Deg} L(\mathbf{1}_m)$ (See Section 1). Main results are Theorem 1.2 and Corollary 2.3. We prove them by using Gindikin gamma function on a positive Hermitian cone in Section 2. On the other hand, the representation $L(\mathbf{1}_m)$ is realized on the so-called determinantal variety, and the calculation of $\operatorname{Deg} L(\mathbf{1}_m)$ is equivalent to obtaining the degree of the determinantal variety. Its degree is already known as Giambelli's formula and proved by Harris and Tu [4] with the help of Thom-Porteous formula. Therefore our formula gives an alternative proof of the Giambelli's formula. We shall explain it briefly in Section 3.

1. Bernstein degree of $L(\mathbf{1}_m)$. Let K be a maximal compact subgroup of G. Then K is isomorphic to the non-trivial double cover of $U(n)$. K-finite vectors in $\omega^{\otimes m}$ can be identified with $\det^{m/2} \otimes \mathbf{C}[M_{n,m}]$ by the Fock realization of ω, where $M_{n,m}$ denotes the space of $n \times m$ matrices. In this picture, K acts naturally from the left (but with the shift by $\det^{m/2}$) and $O(m)$ acts from the right. By the characterization of $L(\mathbf{1}_m)$, we get

$$L(\mathbf{1}_m)\big|_K \simeq \det{}^{m/2} \otimes \mathbf{C}[M_{n,m}]^{O(m)}.$$

The following lemma is well-known. See [5, p. 35], for example.

Lemma 1.1. *As a representation of $U(n)$, we have the multiplicity free decomposition*

$$\mathbf{C}[M_{n,m}]^{O(m)} \simeq \bigoplus_{l(\lambda) \le m} \tau_{2\lambda},$$

where τ_μ denotes the irreducible finite dimensional representation of $U(n)$ with the highest weight μ, and the summation is taken over all the partition λ of the non-negative integers of length less than or equal to m.

Using this lemma, we can define a natural K-invariant filtration of $L(\mathbf{1}_m)$ by putting $L(\mathbf{1}_m)_k = \det^{m/2} \otimes \left(\bigoplus_{|\lambda| \le k, l(\lambda) \le m} \tau_{2\lambda} \right)$ $(k \ge 0)$. Let $d = \operatorname{Dim} L(\mathbf{1}_m)$ be the *Gelfand-Kirillov dimension* of $L(\mathbf{1}_m)$ and denote by $\operatorname{Deg} L(\mathbf{1}_m)$ the *Bernstein degree* (see [10] for definition). Then the theory of Hilbert polynomials tells us that, for sufficient large k, $\dim L(\mathbf{1}_m)_k$ is a polynomial in k and the top term is given by

$$\dim L(\mathbf{1}_m)_k = \frac{\operatorname{Deg} L(\mathbf{1}_m)}{d!} k^d + (\text{lower terms in } k).$$

It is easy to see that $d = \operatorname{Dim} L(\mathbf{1}_m) = nm - m(m-1)/2$ (cf. Eq. (1) below).

Theorem 1.2. *The Bernstein degree of $L(\mathbf{1}_m)$ is given by*

$$\operatorname{Deg} L(\mathbf{1}_m) = \frac{2^{d-m} d!}{m! \prod_{i=1}^m (n-i)!} \times \int_{x_i \ge 0, \sum_{i=1}^m x_i \le 1} (x_1 x_2 \cdots x_m)^{n-m} \times \prod_{1 \le i < j \le m} |x_i - x_j| dx_1 dx_2 \cdots dx_m.$$

Remark 1.3. We shall give the exact formula for the integral in the next section.

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