On the homology of Torelli groups and Torelli spaces

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The purpose of the present note is to announce our recent results concerning of the rational homology of Torelli groups, Torelli spaces, and certain subgroups of Torelli groups. The detail will appear elsewhere. Let Σ_g be a closed orientable surface of genus $g \geq 2$. Let $\Gamma^n_{g,r}$ be the mapping class group of Σ_g relative to n marked points and r embedded disks. Namely it is the group of connected components of all the orientation-preserving diffeomorphisms of Σ_g which fix marked points and embedded disks pointwise. The action of $\Gamma^n_{g,r}$ on the first integral homology of Σ_g induces a surjective homomorphism

$$\Gamma^n_{g,r} \to Sp(2g, \mathbf{Z}),$$

where $Sp(2g, \mathbf{Z})$ is the Siegel modular group of degree g. The *Torelli group* $\mathcal{I}^n_{g,r}$ is defined to be its kernel so that we have an extension

$$1 \to \mathcal{I}^n_{g,r} \to \Gamma^n_{g,r} \to Sp(2g, \mathbf{Z}) \to 1.$$

Henceforth we omit the decorations n and r when they are zero (e.g. $\mathcal{I}^n_{g,0} = \mathcal{I}^n_g$ and $\mathcal{I}^0_{g,r} = \mathcal{I}_{g,r}$). The Torelli groups are related to characteristic classes of surface bundles as well as the Casson invariants and the finite type invariants of homology 3-spheres (see [2, 13, 14, 15] for instance).

Let $\mathcal{T}^n_{g,r}$ be the Teichmüller space of genus g with $n + r$ marked points and r tangent vectors. Namely it is the space of all conformal structures on Σ_g up to isotopies that fix $n + r$ marked points $\{p_1, \cdots, p_{n+r}\}$ pointwise and act trivially on the tangent space $T_{p_i}\Sigma_g$ for $n+1 \leq i \leq n+r$. It is a complex manifold of dimension $3g - 3 + n + 2r$. The mapping class group $\Gamma^n_{g,r}$ acts on $\mathcal{T}^n_{g,r}$ properly discontinuously and the quotient space $\mathcal{T}^n_{g,r}/\Gamma^n_{g,r}$ is identified with the moduli space of curves of genus g with $n + r$ marked points and r tangent vectors. The Torelli group $\mathcal{I}^n_{g,r}$ is torsion-free and hence it acts on $\mathcal{T}^n_{g,r}$ freely so that the quotient space $T^n_{g,r} = \mathcal{T}^n_{g,r}/\mathcal{I}^n_{g,r}$ is a complex manifold which is called the *Torelli space*. The Torelli space admits a moduli interpretation as well (see [3, 5]). Moreover, since $\mathcal{T}^n_{g,r}$ is contractible, $T^n_{g,r}$ is the classifying space of $\mathcal{I}^n_{g,r}$ so that there is a canonical isomorphism

$$H_*(\mathcal{I}^n_{g,r}, \mathbf{Z}) \cong H_*(T^n_{g,r}, \mathbf{Z}).$$

Torelli spaces are related to the study of the moduli spaces of curves with or without level structure (see [3, 4, 5] for instance).

Little is known about the structure of the homology of Torelli groups and Torelli spaces apart from D. Johnson's several fundamental results obtained in a series of papers [6, 7, 8, 9] (see also [10, 5]). In particular, he proved that $\mathcal{I}_g$ and $\mathcal{I}_{g,1}$ are finitely generated when $g \geq 3$ and computed their first integral homology. As a consequence of his result, $\mathcal{I}^n_{g,r}$ is finitely generated for all $g \geq 3$ and $n, r \geq 0$ (see [3, 5] for the explicit computation of the first integral homology of $\mathcal{I}^n_{g,r}$). On the contrary, A. Miller and D. McCullough [12] showed that $\mathcal{I}_2$ is not finitely generated. G. Mess [11] showed that $\mathcal{I}_2$ is a free group on infinitely many generators. D. Johnson and J. Millson showed that $H_3(\mathcal{I}_3, \mathbf{Z})$ contains a free abelian group of infinite rank (cf. [11]). It is known that $H_2(\mathcal{I}_g, \mathbf{Q})$ and $H_2(\mathcal{I}^1_g, \mathbf{Q})$ are nontrivial for $g \geq 3$ (see [14, 4] for precise statements). To date, it is not known whether $\mathcal{I}^n_{g,r}$ is finitely presented for $g \geq 3$. For other results concerning of the homology of Torelli groups and Torelli spaces, see [10, 14, 11, 5] and references therein. We have proved:

Theorem 1. *For $g \geq 7$ and $n, r \geq 0$, the rational homology $H_*(\mathcal{I}^n_{g,r}, \mathbf{Q})$ of the Torelli group $\mathcal{I}^n_{g,r}$ is infinite dimensional over* $\mathbf{Q}$.

Corollary. *For $g \geq 7$ and $n, r \geq 0$, the rational homology $H_*(T^n_{g,r}, \mathbf{Q})$ of the Torelli space $T^n_{g,r}$ is infinite dimensional over* $\mathbf{Q}$.

In other words, we have $\dim_{\mathbf{Q}} H_i(\mathcal{I}^n_{g,r}, \mathbf{Q}) = \dim_{\mathbf{Q}} H_i(T^n_{g,r}, \mathbf{Q}) = \infty$ for some $i > 1$, since $T^n_{g,r}$ is a finite dimensional complex manifold. D. Johnson [10] asked whether the Torelli space T^1_g is homotopy

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