On the units and the class numbers of certain composita of two quadratic fields

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1. Preliminaries. Let k_1 be a real quadratic field and ε_1 (>1) be the fundamental unit of k_1. We shall fix a unit $\eta_1 = \varepsilon_1^{2i+1}$, which is an odd power of the fundamental unit ε_1 with $i \geq 0$. Then there exists some positive integer M such that η_1 is written in the form

$$\eta_1 = \frac{M + \sqrt{M^2 \pm 4}}{2}.$$

Let $\bar{\eta}_1$ be the field conjugate of η_1. Put $D = M^2 \pm 4$. Then D is not necessarily square-free, and we denote the square-free part of D by D_0. When we use the notation $\pm y$ or $\mp z$, $+y$ and $-z$ correspond to the upper case $D = M^2 + 4$, which will be called the *plus case*, and $-y$ and $+z$ correspond to the lower case $D = M^2 - 4$, which will be called the *minus case*.

Put

$$g_n = \eta_1^n + \bar{\eta}_1^n, \qquad h_n = \frac{\eta_1^n - \bar{\eta}_1^n}{\sqrt{D}}.$$

Then the sequences $\{g_n\}_{n\in\mathbf{N}}$ and $\{h_n\}_{n\in\mathbf{N}}$ are the non-degenerated second order linear recurrence sequences defined by

$$g_{n+2} = Mg_{n+1} \pm g_n, \quad h_{n+2} = Mh_{n+1} \pm h_n,$$

with the initial terms $g_0 = 2$, $g_1 = M$ and $h_0 = 0$, $h_1 = 1$.

The purpose of this note is to report our results on the class number h_K and the unit group E_K of the biquadratic field $K = \mathbf{Q}(\sqrt{D}, \sqrt{h_{2n+1}^2 - 1})$: see Theorems 1 and 2. Only sketches of proofs will be provided and details will be published elsewhere.

For any a, $b \in \mathbf{Z} \setminus \{0\}$, we put $a \sim b$ if and only if ab is a perfect square. So

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \sim \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \Longleftrightarrow a_1 \sim a_2 \text{ and } b_1 \sim b_2.$$

Moreover, $M^2 - D = \mp 4$ and $g_{2n+1}^2 - Dh_{2n+1}^2 = \mp 4$ imply

$$g_{2n+1}^2 - M^2 = D(h_{2n+1}^2 - 1).$$

Then we shall verify that $h_{2n+1}^2 - 1 \not\sim 1$ and $h_{2n+1}^2 - 1 \not\sim D$ except for finitely many indices n. So except for finitely many indices n, we will construct a family of real bicyclic biquadratic fields

$$K = \mathbf{Q}\left(\sqrt{D}, \sqrt{h_{2n+1}^2 - 1}\right) \quad (n \geq 1).$$

Then K has three subfields:

$$k_1 = \mathbf{Q}\left(\sqrt{D}\right), \quad k_2 = \mathbf{Q}\left(\sqrt{h_{2n+1}^2 - 1}\right),$$
$$k_3 = \mathbf{Q}\left(\sqrt{g_{2n+1}^2 - M^2}\right).$$

We have a unit η_2 in k_2 defined by

$$\eta_2 = h_{2n+1} + \sqrt{h_{2n+1}^2 - 1},$$

and we will denote by ε_2 the fundamental unit of k_2.

Concerning the recurrence sequence $\{g_n\}_{n\in\mathbf{N}}$, one can verify $M | g_{2n+1}$ by induction. So we also have a unit η_3 in $k_3 = \mathbf{Q}(\sqrt{(g_{2n+1}/M)^2 - 1})$, namely

$$\eta_3 = g_{2n+1}/M + \sqrt{(g_{2n+1}/M)^2 - 1},$$

and we will denote by ε_3 the fundamental unit of k_3.

Let E be the group $\langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$. Then the group index $[E_K : E]$ is called the unit index of K and is known to be $1, 2$ or 4 in general. Let us quote a result of Shorey-Stewart [14].

Lemma 1. *Let d be an integer > 1. Then there exists a constant C_1, which is effectively computable in terms of M and d such that for any $n \geq C_1$,*

$$g_n \not\sim d \quad \text{and} \quad h_n \not\sim d.$$

Let us list several properties of the above two linear recurrences $\{g_n\}_{n\in\mathbf{N}}$ and $\{h_n\}_{n\in\mathbf{N}}$.

Proposition 1. *For any index $n \geq 0$,*

(i) $h_{2n+1} + (\mp 1)^n = g_n h_{n+1}$,
(ii) $h_{2n+1} - (\mp 1)^n = g_{n+1} h_n$,
(iii) $g_{2n+1} + (\mp 1)^n M = g_n g_{n+1}$,

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