# Valuation of default swap with affine-type hazard rate

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**1. Introduction.** The aim in this paper is to give implements applicable to valuing "default swap", a kind of financial commodity called credit derivative. Davis and Marvoidis [2], under the assumption that the hazard rate is a Gaussian and independent of the riskless spot rate, evaluated the value of the swap by using forward measure approach and the integral approximation. The Gaussian hazard rate model has, however, an undesirable property that it may become negative, hence, the probability of not-default at some time may be over one. So the author is motivated by the idea that CIR term structure model, for example, must be effective for modeling hazard rate.

The main result concerns the formula which computes the expectation of the special functional of the hazard rate, under the assumption that the hazard rate process follows the so-called affine type model including CIR model. It is proved by using Itô formula and usual calculus.

**2. The result.** Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Denote by $B$ a one dimensional standard Brownian motion on the above space.

**Theorem 1.** *Let* $T \in (0, \infty)$.

*Let $h_t$ satisfy the following stochastic differential equation (called the affine-type model) on* $[0, T]$.

$$(1) \qquad dh_t = m(h_t, t)dt + \sigma(h_t, t)dB_t, \quad h_0 > 0,$$

*where $m$ and $\sigma$ are deterministic functions of the following form*:

$$m(x,t) = m_1(t) + m_2(t)x, \sigma(x,t)^2 = \sigma_1(t) + \sigma_2(t)x$$

*for deterministic functions $m_i(t), \sigma_i(t)$ $(i = 1, 2)$ with $\sigma_2 \neq 0$ and*

$$(2) \qquad m_1(t) - m_2(t)\sigma_1(t)\sigma_2(t)^{-1} \geq 0, t \in [0, T].$$

*Let $\beta$ be a nonnegative real number and $\kappa(t)$ be a strictly positive deterministic continuously differentiable function.*

*Then the following equality holds*: *for* $t \in [0, T]$,

$$E[\exp(-\int_0^t \kappa(s)h_s ds - \beta h_t)h_t] = \Phi(t)\frac{G(t) + J(t)h_0}{K(t)}$$

*where*

$$\Phi(t) = \exp(-a(t) - b(t)h_0),$$
$$G(t) = -\frac{1}{2}\sigma_1(t)b(t)^2 + (b(t) - \beta)m_1(t),$$
$$J(t) = -\frac{1}{2}\sigma_2(t)b(t)^2 + m_2(t)b(t) + \kappa(t),$$
$$K(t) = \kappa(t) + \beta m_2(t) - \frac{1}{2}\beta^2\sigma_2(t),$$

*and $a(t), b(t)$ are solutions to the following differential equations*:

$$(3) \qquad \begin{cases} b'(t) = -\frac{1}{2}\sigma_2(t)b(t)^2 + m_2(t)b(t) + \kappa(t) \\ a'(t) = -\frac{1}{2}\sigma_1(t)b(t)^2 + m_1(t)b(t) \\ a(0) = 0, \ b(0) = \beta. \end{cases}$$

**Remark.** The condition (2) guarantees the existence of a solution $h$ to the SDE (1) with $h_t \geq -\sigma_1(t)\sigma_2(t)^{-1}$ for all $t \in [0, T]$. (See Duffie [3].) In particular, by assuming $\sigma_1 = 0$, the positive solution is achieved.

To begin with, we state the following crucial proposition without proof.

**Proposition 2.** *Assume $h_t$ satisfies* (1) *in Theorem* 1.

*For any non-negative $\beta$ and strictly positive function $\kappa(t)$, we have*

$$E[\exp(-\beta h_t - \int_0^t \kappa(s)h_s ds)] = \exp(-a(t) - b(t)h_0),$$

*where $a(t)$ and $b(t)$ are solutions of* (3).

It goes without saying that if $a(t)$ and $b(t)$ have an explicit form as a function of $\beta$ (see Example 3), the result in the theorem can be easily achieved by differentiating $\exp(-a(t) - b(t)h_0)$ in $\beta$. Now we give the proof of Theorem 1 applicable to other general cases.

*Proof.* Let $a$ and $b$ be solutions to (3). Now

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