Exit probability of two-dimensional random walk from the quadrant

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1. Introduction and preliminaries. Let

$$Z_0 = 0, Z_1 = (X_1, Y_1), Z_2 = (X_2, Y_2), ...$$

be a random walk in the two-dimensional integer lattice $\boldsymbol{Z}^2$. By a random walk we mean a stochastic sequence with stationary independent increments starting at the origin. Throughout the paper we impose on the random walk the following assumptions.

Assumption 1.1. *For every* $\boldsymbol{\theta} = (\theta_1, \theta_2)$ *in* $\boldsymbol{R}^2$,

$$\lambda(\boldsymbol{\theta}) := E(e^{\boldsymbol{\theta}\cdot Z_1}) < \infty,$$

where $\boldsymbol{\theta}\cdot\boldsymbol{z}$ *denotes the inner product in* $\boldsymbol{R}^2$.

Let D_i $(i = 1, 2, 3, 4)$ be the *i th quadrant* in $\boldsymbol{R}^2$, that is,

$$D_1 = \{(x, y) \in \boldsymbol{R}^2 | x > 0, y > 0\},$$
$$D_2 = \{(x, y) \in \boldsymbol{R}^2 | x < 0, y > 0\},$$
$$D_3 = \{(x, y) \in \boldsymbol{R}^2 | x < 0, y < 0\},$$

and

$$D_4 = \{(x, y) \in \boldsymbol{R}^2 | x > 0, y < 0\}.$$

Assumption 1.2. $\boldsymbol{\mu} = E(Z_1) \in D_1$, *and* $P(Z_n \in D_4) > 0$ *for some positive integer n.*

Assumption 1.3. *The y-coordinate of the random walk is left-continuous, that is,* $P(Y_1 \in \{-1, 0, 1, 2, ...\}) = 1$.

Let a and b be positive integers. In this paper we will take a arbitrarily fixed, so we omit a in many of our statements and notations. Set

$$T_b := \inf\{n \geq 0 | (a, b) + Z_n \notin D_1\}$$

$(\inf \emptyset = \infty)$. Define

$$D_4^* := \{(x, y) | x > 0,\ y \leq 0\}$$

and

$$r_b := P(T_b < \infty, (a, b) + Z_{T_b} \in D_4^*).$$

Since $Z_n \sim \boldsymbol{\mu} n$ *a.s.* $(n \to \infty)$ by the *strong law of large numbers*, we have $r_b \to 0$ $(b \to \infty)$ from the first condition of Assumption 1.2. The purpose of this paper is to study the decay rate of r_b to 0. Our problem is a two-dimensional extension of the asymptotic analysis of *ruin probability* for one dimensional random walk with positive drift.

Let Θ denote the contour of the moment generating function $\lambda(\boldsymbol{\theta})$ at the level 1, that is, $\Theta = \{\boldsymbol{\theta} \in \boldsymbol{R}^2 | \lambda(\boldsymbol{\theta}) = 1\}$. It is shown from Assumptions 1.1 and 1.2 the following lemma. (See, e.g., Ney *et al.* [4]).

Lemma 1.1. *Θ is a smooth convex curve. Moreover, it intersects the θ_2-axis at two points; the one is the origin and the other is* $\widetilde{\boldsymbol{\theta}} = (0, \widetilde{\theta}_2)$ *with* $\widetilde{\theta}_2 < 0$.

Note that, if $\boldsymbol{\theta} \in \Theta$, *then* $\exp(\boldsymbol{\theta}\cdot\boldsymbol{z})$ *is a harmonic function of the random walk, namely, it satisfies*

$$E(\exp\{\boldsymbol{\theta}\cdot(Z_1 + \boldsymbol{z})\}) = \exp(\boldsymbol{\theta}\cdot\boldsymbol{z}) \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{R}^2.$$

From now on we always take $\boldsymbol{\theta}$ as an element of Θ. We will not indicate it in our statements. Let $F(\boldsymbol{z}) := P(Z_1 = \boldsymbol{z})$ and introduce a new probability function on $\boldsymbol{Z}^2$ by

$$F^{(\boldsymbol{\theta})}(\boldsymbol{z}) := \exp(\boldsymbol{\theta}\cdot z)F(\boldsymbol{z}).$$

By $P^{(\boldsymbol{\theta})}$ we denote the probability measure of the random walk with the one-step probability function $F^{(\boldsymbol{\theta})}(\boldsymbol{z})$. By elementary observation we get the following formulas and lemma:

$$\boldsymbol{\mu}^{(\boldsymbol{\theta})} := E^{(\boldsymbol{\theta})}(Z_1) = \nabla\lambda(\boldsymbol{\theta}). \tag{1.1}$$

Lemma 1.2. *The following two statements are equivalent*:
(i) $P^{(\boldsymbol{\theta})}(T_b < \infty) = 1$. (ii) $\boldsymbol{\mu}^{(\boldsymbol{\theta})} \notin D_1$.

Put

$$\eta_b(\boldsymbol{\theta}) := 1(T_b < \infty, (a, b) + Z_{T_b} \in D_4^*) \times \exp(-\boldsymbol{\theta}\cdot Z_{T_b}), \tag{1.2}$$

where $1(A)$ is the indicator function of an event A, that is, $1(A) = 1$ if A occurs and $1(A) = 0$ otherwise. Then, as is shown in Lehtonen et al. [2], we have

$$r_b = E^{(\boldsymbol{\theta})}(\eta_b(\boldsymbol{\theta})). \tag{1.3}$$

As will be discussed in §§ 2 and 3, our key observation on the problem is the following: '*To choose the* $\boldsymbol{\theta}$ *from* Θ *which is most preferable to get an asymptotic formula for* r_b $(b \to \infty)$ *via* (1.3)'. The obser-