Modified complexity and $*$-Sturmian word

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We give analogies of the complexity $p(n)$ and Sturmian words which are called the $*$-complexity $p_*(n)$ and $*$-Sturmian words. We announce theorems about $*$-Sturmian words in this paper. The proofs and details will be published elsewhere. We consider words over an alphabet $L = \{0, 1\}$. Let L^n be the set of all words of length $n \geq 0$, $L^0 = \{\lambda\}$, λ is the empty word. Let L^* be the set of all finite words and $L^{\boldsymbol{N}}$ (resp. $L^{-\boldsymbol{N}}$) be the set of right-sided (resp. left-sided) infinite words. A two-sided infinite words $W \in L^{\boldsymbol{Z}}$ is defined to be a map $W : \boldsymbol{Z} \to L$. We identify two words $V, W \in L^{\boldsymbol{Z}}$ if $V(x+y) = W(x)$ for all $x \in \boldsymbol{Z}$ for some fixed $y \in \boldsymbol{Z}$. We put $L^\wedge = L^* \cup L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}$. We denote the set of all subwords of W by $D(W)$. We put $D(n; W) := D(W) \cap L^n$ ($n \geq 0$). The complexity of a word W is a function defined by

$$p(n) = p(n; W) := \sharp D(n; W).$$

A $*$-subword w of W is a word $w \in D(W)$ which occurs infinitely many times in W. We put $D_*(n; W) := D_*(W) \cap L^n$, where $D_*(W)$ is the set of $*$-subwords of W. We define $*$-complexity

$$p_*(n) = p_*(n; W) := \sharp D_*(n; W).$$

A Sturmian word is defined to be a word $W \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}$ satisfying

$$|\xi(A) - \xi(B)| \leq 1$$

for any $A, B \in D(n; W)$ for all $n \geq 0$, where $\xi(w)$ denotes the number of occurrences of a symbol 1 appearing in a word $w \in L^*$, cf. [2]. We define a $*$-Sturmian word to be a word $W \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}$ satisfying

$$|\xi(A) - \xi(B)| \leq 1$$

for any $A, B \in D_*(n; W)$ for all $n \geq 0$. Let $\sigma(n; W) = \max_{A \in D(n;W)} \xi(A)$ and $\sigma'(n; W) = \min_{A \in D(n;W)} \xi(A)$.

Theorem 1 (Morse and Hedlund [2]). *If W is a Sturmian word, then $p(n; W) \leq n + 1$, and there is the density* $\alpha = \lim_{n\to\infty} \dfrac{\sigma(n, W)}{n} = \lim_{n\to\infty} \dfrac{\sigma'(n, W)}{n}$.

We can classify one-sided or two-sided infinite Sturmian words as follows:

(Type I) α is irrational,

(Type II) α is rational and W is purely periodic,

(Type III) α is rational and W is not purely periodic.

It is known that each case can occur. The words of Type III will be referred to as skew Sturmian words. Let $0 \leq \alpha \leq 1$ and β be real numbers. We define $G(n, \alpha, \beta) = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$ and $G'(n, \alpha, \beta) = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$, where $\lfloor x \rfloor$ is the greatest integer which does not exceed x and $\lceil x \rceil$ is the least integer which is not smaller than x. A word $G(\alpha, \beta) \in L^{\boldsymbol{N}}$ is defined by

$$G(\alpha, \beta) = G(0, \alpha, \beta) G(1, \alpha, \beta) \cdots G(n, \alpha, \beta) \cdots.$$

$G'(\alpha, \beta)$ is defined similarly by using $G'(n, \alpha, \beta)$. We set $G(\alpha) = G(\alpha, 0)$, $G'(\alpha) = G'(\alpha, 0)$, $G(n, \alpha) = G(n, \alpha, 0)$ and $G'(n, \alpha) = G'(n, \alpha, 0)$.

Theorem 2 (Morse and Hedlund [2]). *If α is irrational* (*resp. rational*), *then $G(\alpha, \beta)$ and $G'(\alpha, \beta)$ are Sturmian words of Type I* (*resp. TypeII*). *Conversely, if $W \in L^{\boldsymbol{N}}$ is a Sturmian word of type I with density* $\alpha = \lim_{n\to\infty} \dfrac{\sigma(n, W)}{n}$, *there exists a real number β such that $W = G(\alpha, \beta)$ or $W = G'(\alpha, \beta)$.*

For A, $B \in L^*$ we denote by $\{A, B\}^*$ the set

$$\{A, B\}^* := \{w_1 \cdots w_n; w_i = A \text{ or } B \; n \geq 0\}.$$

We say a word $W \in \{a, b\}^*$ is strictly over $\{a, b\}$ if both a and b eventually occur in W. w^* (resp. ${}^*w$) ($\lambda \neq w \in L^*$) denote the words $w^* := www \cdots \in L^{\boldsymbol{N}}$ (resp. ${}^*w := \cdots www \in L^{-\boldsymbol{N}}$), w^n ($n \in \boldsymbol{N} \cup \{0\}$, $w \in L^*$) is the word $w^n := v_1 v_2 \cdots v_n$ ($v_i = w$). We mean by ${}^*vw$ (resp. vw^*) the word $({}^*v)w$ (resp. $v(w^*)$).

Theorem 3 (Morse and Hedlund [2]). *Let $W \in L^{\boldsymbol{N}}$ be a purely periodic Sturmian word with*

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