On a naturality of Chern-Mather classes

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Introduction. The Chern-Schwartz-MacPherson class (or more precisely natural transformation) is the unique natural transformation from the covariant constructible function functor to the covariant integral homology functor satisfying the normalization that the value of the characteristic function of a nonsingular compact complex analytic variety is equal to the Poincaré dual of the total Chern cohomology class of the tangent bundle. The existence of such a transformation was conjectured by Deligne and Grothendieck and was proved by MacPherson [10]. The novelity of MacPherson's proof is introducing the notion of local Euler obstruction (which was independently introduced by Kashiwara [7] also) and assigning the Chern-Mather class to this local Euler obstruction, not to the characteristic function. Although the Chern-Mather class is a very geometrically simple homology class, the assignment of the Chern-Mather class to the characteristic function does not give such a natural transformation.

It is often said that few "functorial" properties are know for the Chern-Mather class (e.g., see [6, *Note*, page 377, right after Example 19.1.7]), although the assignment of the Chern-Mather class to the local Euler obstruction is perfectly "natural", which is the main part of MacPherson's proof. In this paper, using this fine naturality of this assingment, we interpret the Chern-Mather class in the same way as the Chern-Schwartz-MacPherson class. Furthermore, by introducing the notion of a "q-deformed" local Euler obstruction which unifies the characteristic function and the local Euler obstruction, we give a "q-deformed" Chern-Schwartz-MacPherson class natural transformation, which specializes to the Chern-Mather class natural transformation for $q = 0$ and the Chern-Schwartz-MacPherson class natural transformation for $q = 1$.

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1. Constructible functions and Chern-Schwartz-MacPherson classes. Let $\mathcal{F}(X)$ denote the abelian group of constructible functions on X. The correspondence $\mathcal{F}$ assigning to each variety X the abelian group $\mathcal{F}(X)$ becomes a covariant functor when we consider the following "geometrically defined" pushforward:

$$\big(f_*\mathbb{1}_W\big)(y) := \chi\big(f^{-1}(y) \cap W\big),$$

which is linearly extended with respect to the generators $\mathbb{1}_W$ (see [10, 12]).

For the algebraic category Deligne and Grothendieck conjectured and MacPherson proved:

Theorem (1.1) (MacPherson's theorem [10])**.** *For the covariant functors $\mathcal{F}$ and H_* there exists a unique natural transformation*

$$C_* : \mathcal{F} \to H_*$$

satisfying (normalization condition) that for X nonsingular

$$C_*(\mathbb{1}_X) = c(TX) \cap [X],$$

where $c(TX)$ is the total Chern cohomology class of the tangent bundle TX and $[X]$ is the fundamental homology class of X.

MacPherson first observed that the abelian group of constructible functions are freely generated by local Euler obstructions of the closed subvarieties and via the graph construction method he proved that the association of the Chern-Mather class $C^M(W)$ to the local Euler obstruction Eu_W;

$$C_* : \mathrm{Eu}_W \longmapsto C^M(W),$$

not to the characteristic function $\mathbb{1}_W$, is natural, i.e.,

$$f_*C_*(\mathrm{Eu}_W) = C_*(f_* \mathrm{Eu}_W). \tag{1.2}$$