On the topology of the moduli space of negative constant scalar curvature metrics on a Haken manifold

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1. Introduction. The topology of the space of positive scalar curvature metrics on a closed manifold M has been studied by several authors ([6]). It turns out that the topology of this space is very complicated, and the moduli space of positive scalar curvature metrics quotient by the diffeomorphism group of M can have infinitely many connected components. By contrast, the topology of the space of negative scalar curvature metrics is very simple ([7]).

Let M be a closed connected manifold. Denote by $\mathcal{M}_{-1}(M)$ the set of all Riemannian metrics with scalar curvature -1. The diffeomorphism group acts on $\mathcal{M}_{-1}(M)$ by pull-back. In this paper, we will report the topological structure of the moduli space $\mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$, the space of Riemannian metrics with scalar curvature -1 devided by the group $\mathrm{Diff}_0(M)$ of diffeomorphisms which are isotopic to the identity map. The result gives a fact that if M is a closed connected Haken manifold with no nontrivial symmetry, then the moduli space $\mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$ is a contractible manifold. Note that this result is an analogue to the contractibility of the Teichmüller space on an oriented surface with negative Euler number ([3], [10]). It seems that there are similarities between Haken manifolds and oriented surfaces with non-positive Euler number.

2. The space of negative constant scalar curvature metrics. Let M be a closed n-manifold, and $\mathcal{M}(M)$ be the space of all Riemannain metrics on M. For $g \in \mathcal{M}(M)$, let R_g denote the scalar curvature of g, and $\mathcal{M}_{-1}(M)$ denote the space of Riemannian metrics with scalar curvature -1. It is known that if M is a closed n-manifold, $n \geq 3$, then M admits a Riemannian metric with scalar curvature -1, i.e., $\mathcal{M}_{-1}(M)$ is a non-empty set if $\dim M \geq 3$. We denote by $L^2_k\mathcal{M}(M)$ the space of all L^2_k-metrics, where L^2_k is a Sobolev space whose derivatives of order less than or equal to k are all in L^2. Then the space $L^2_k\mathcal{M}(M)$ is a Hilbert manifold for $2k > n$. It is known that the space $\mathcal{M}(M)$ is an ILH-manifold in the sense of the inverse limit of Hilbert manifolds: $\mathcal{M}(M) = \lim_{\leftarrow} L^2_k\mathcal{M}(M)$ ([8]).

For $2k > n+2$, let $\mathcal{R} : L^2_k\mathcal{M}(M) \to L^2_{k-2}(M)$ defined by $\mathcal{R}(g) := R_g$ denote the scalar curvature map. The tangent space at $g \in L^2_k\mathcal{M}(M)$ can be identified with the space $L^2_k(M; S^2T^*M)$ of symmetric $(0,2)$-tensor fields of class L^2_k. We denote its differential at $g \in L^2_k\mathcal{M}(M)$ by $\beta_g := d\mathcal{R}_g : L^2_k(M; S^2T^*M) \to L^2_{k-2}(M)$.

Lemma 2.1. *The differential β_g of the scalar curvature map is given by*

$$\beta_g(h) = -\Delta_g(\mathrm{tr}_g h) + \delta_g\delta_g h - (h, \mathrm{Ric}_g),$$

where δ_g is the formal adjoint of the covariant derivative of g and Ric_g is the Ricci curvature of g.

Theorem 2.2 ([2])**.** *Let $g \in L^2_k\mathcal{M}(M)$, $2k > n+2$, with $R_g = -1$. Then β_g is surjective.*

Theorem 2.3. *$\mathcal{M}_{-1}(M)$ is a smooth contractible ILH-submanifold of $\mathcal{M}(M)$ with tangent space $T_g\mathcal{M}_{-1}(M)$ at $g \in \mathcal{M}_{-1}(M)$ given as $\mathrm{Ker}\,\beta_g$ the kernel of the differential of the scalar curvature map.*

3. Some results on Haken manifolds. A compact connected orientable 3-manifold M is said to be *irreducible* if every 2-sphere S^2 in M bounds a 3-ball B^3.

Let M be a compact connected orientable 3-manifold. Let S be a compact connected orientable surface, and let $i : S \to M$ be an embedding of S into M. Then i induces a homomorphisms on the homotopy groups $i_* : \pi_k(S) \to \pi_k(M)$ for $k \geq 1$. The embedded surface $i(S)$ is *incompressible* if the induced homomorphism i_* is injective on the fundamental group $\pi_1(S)$. A 3-manifold is *sufficiently large* if it contains an incompressible surface of genus greater than zero.

Definition 3.1. *A Haken manifold M is an irreducible compact connected orientable sufficiently large 3-manifold.*

Remark 3.2. A connected manifold M is called a $K(\pi,1)$-*manifold* if the fundamental group