Quasilinear degenerate elliptic equations with absorption term

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1. Introduction. Let $N \geq 1$ and $p > 1$. Let F be a given **smooth compact set** and Ω be a bounded open set of $\mathbf{R}^N$ satisfying $F \subset \Omega \subset \mathbf{R}^N$ and $F \neq \phi$. We also set $\Omega' = \Omega \setminus \partial F$, where $\partial F = F \setminus \overset{\circ}{F}$. Here by $\overset{\circ}{F}$ we denote the interior of F, which may be empty.

By $H^{1,p}(\Omega)$ we denote the space of all functions on Ω, whose generalized derivatives $\partial^\gamma u$ of order ≤ 1 satisfy

$$(1\text{-}1) \quad ||u||_{1,p} = \sum_{|\gamma| \leq 1} \left(\int_\Omega |\partial^\gamma u(x)|^p \, dx \right)^{1/p} < +\infty,$$

and by $H^{1,p}_{\text{loc}}(\Omega)$ the local version of $H^{1,p}(\Omega)$. For $u \in H^{1,p}_{\text{loc}}(\Omega')$, we define a generalized p-harmonic operator by

$$(1\text{-}2) \qquad L_p u = -\operatorname{div}(A(x)|\nabla u|^{p-2}\nabla u),$$

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$, and $A(x) \in C^1(\Omega')$ is positive in $\Omega \setminus F$ and vanishes in $\overset{\circ}{F}$. We shall study the Dirichlet boundary problem for the genuinely degenerate elliptic operators L_p with absorption term:

$$(1\text{-}3) \quad \begin{cases} L_p u + B(x)Q(u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $B(x)$ is a nonnegative function on Ω, and $Q(t)$ is a continuous and strictly monotone increasing function on $\mathbf{R}$. In connection with this problem we shall treat two topics in the present paper. Namely, one is concerned with removable singularities of solutions for (1-3) and the other is the unique existence property of bounded solutions. We note that if $p = 2$, then these topics were already treated in the author's paper [6] under a similar framework. By virtue of Kato's inequality and a maximum principle, the unique existence of bounded solutions was established. Since Kato's inequality does not work effectively in the quasilinear case, we shall employ in this paper a priori estimates, a comparison principle and a weak maximum principle instead. Since the operators L_p are rather general, we need to modify them suitably so that they are applicable to our problems.

2. Preliminaries. In this section we prepare our basic framework and some notations which are of importance through the present paper.

Let $N \geq 1$ and $p > 1$. Let F and Ω be a nonempty smooth compact set and bounded open suset of $\mathbf{R}^N$ respectively, satisfying $F \subset \Omega$, and set $\Omega' = \Omega \setminus \partial F$. Here ∂F is defined as $\partial F = F \setminus \overset{\circ}{F}$. In the next we define a distance to ∂F.

Definition 1. By $d(x)$ we denote a distance function $d(x) = \operatorname{dist}(x, \partial F)$.

Remark. A distance function $d(x)$ is Lipschitz continuous and differentiable almost everywhere. Moreover one can approximate it by a smooth function. Namely there exists a nonnegative smooth function $D(x) \in C^\infty(\Omega')$ such that

$$(2\text{-}1) \quad \begin{aligned} &C(0)^{-1} \leq \frac{D(x)}{d(x)} \leq C(0), \\ &|\partial^\gamma D(x)| \leq C(|\gamma|) d(x)^{1-|\gamma|}, \quad x \in \Omega', \end{aligned}$$

where γ is an arbitrary multi-index and $C(|\gamma|)$ is a positive number depending on $|\gamma|$. Therefore one can assume that $d(x)$ is smooth as well without loss of generality. (For the construction of $D(x)$, see [9] for example.)

First we assume the following **(H-1)** on nonnegative functions $A(x)$ and $B(x)$.

(H-1)

$$(2\text{-}2) \quad \begin{cases} A(x) \in C^1(\Omega') \cap L^1_{\text{loc}}(\Omega), \\ A(x) = 0 \quad \text{in} \quad \overset{\circ}{F} = F \setminus \partial F, \\ A(x) > 0 \quad \text{in} \quad \Omega \setminus F, \\ B(x) \in L^\infty_{\text{loc}}(\Omega') \cap L^1_{\text{loc}}(\Omega), \\ B(x) > 0 \quad \text{in} \quad \Omega' = \Omega \setminus \partial F. \end{cases}$$

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