On an identity of theta functions obtained from weight enumerators of linear codes*)

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(Communicated by Shokichi IYANAGA, M. J. A., June 23, 1998)

Abstract: D. P. Maher obtained an identity of theta functions by means of the Lee weight enumerators of linear codes over finite fields. S. S. Rangachari used it to prove an identity of theta functions conjectured by S. Ramanujan. In this paper, we consider the linear codes over finite rings, and give a generalization of Maher's identity. As an application, we generalize the result of Rangachari.

1. Introduction. Let a/n and b/n be rational numbers with $|a|, |b| < n$. We define the Jacobi theta function with a characteristic $(a/n, b/n)$ by

$$\vartheta_{a/n,b/n}(z, \tau) = \sum_{m\in Z}\exp\left\{\pi i\left(m+\frac{a}{n}\right)^2\tau + 2\pi i\left(m+\frac{a}{n}\right)\left(z+\frac{b}{n}\right)\right\}$$

for $z \in \boldsymbol{C}$, $\tau \in \boldsymbol{H} = \{z \in \boldsymbol{C} \mid \mathrm{Im}\, z > 0\}$.

In [3, p. 54] Ramanujan stated, without proof, a certain identity involving theta functions. Rangachari [4] has reformulated and proved this identity of Ramanujan in terms of the Jacobi theta functions:

Theorem(Ramanujan's identity). *For n odd,*

$$\sum_{r=-(n-1)/2}^{(n-1)/2} \vartheta_{r/n,0}(z, n\tau)^n = \vartheta_{0,0}(z, \tau)\cdot F(\tau).$$

For n even,

$$\sum_{r=-n/2+1}^{n/2} \vartheta_{r/n,0}(z, n\tau)^n = \vartheta_{0,0}(z, \tau)\cdot F(\tau)$$

with

$$F(\tau) = \sum_{m=0}^{\infty} a_m \exp(\pi i m\tau),$$

$$a_0 = 1,\ a_j = 0\ (0 < j < n-1),\ a_{n-1} = 2n.$$

Moreover Rangachari has proved the following identity of the theta function of the dual root lattice A^*_{n-1} (For the definition of this function, see below. cf. e.g. [5]. See also Remark 1.2 below):

Theorem(Rangachari). *If n is a prime number p, then*

$$F(\tau) = \Theta_{A^*_{p-1}}(p\tau).$$

To prove this theorem, Rangachari has essentially used the identity of Maher [2] between the Jacobi theta functions and theta functions of lattices defined by linear codes. The identity of Maher is obtained from the Lee weight enumerators of linear codes over a finite field $\boldsymbol{F}_p$.

Rangachari has also proved that this identity holds for the case of $n = 4$. He has conjectured that this identity would hold for any integer $n > 1$.

In this paper we extend the notion of linear codes from codes over a field $\boldsymbol{F}_p$ to codes over a ring $\boldsymbol{Z}/m\boldsymbol{Z}\ (m > 1)$, and we define the Lee weight enumerators of linear codes over $\boldsymbol{Z}/m\boldsymbol{Z}$. As a result we extend the identity of Maher, and, by using this identity, we extend the result of Rangachari:

Theorem 1.1. *For any integer n greater than 1, we have*

$$F(\tau) = \Theta_{A^*_{n-1}}(n\tau).$$

Remark 1.2. In Rangachari's paper [4], the theta function $\Theta_{A^*_{p-1}}(\tau) = \sum_{x\in A^*_{p-1}} q^{x\cdot x}$, where $q = \exp(\pi i\tau)$, was defined by using Voronoi's principal form $x\cdot x = p\langle x, x\rangle$. On the other hand, we use the standard Euclidean scalar product $\langle x, x\rangle$. So our result is slightly different from the result of Rangachari. Indeed, if we denote by $\bar{\Theta}_{A^*_{p-1}}(\tau)$ the theta function defined by Rangachari, we have $\bar{\Theta}_{A^*_{p-1}}(\tau) = \Theta_{A^*_{p-1}}(p\tau)$.

2. Preliminaries. A lattice in $\boldsymbol{R}^n$ is a free $\boldsymbol{Z}$-submodule $L = \boldsymbol{Z}e_1 \oplus \cdots \oplus \boldsymbol{Z}e_n$ of $\boldsymbol{R}^n$ with a $\boldsymbol{R}$-independent $\boldsymbol{Z}$-basis $\{e_1, \ldots, e_n\}$. We use the standard Euclidean scalar product $\langle x, y\rangle = \sum x_i y_i$ of $x, y \in \boldsymbol{R}^n$.

Let L be a lattice with basis $\{e_1, \ldots, e_n\}$, and let A be the Gram matrix $(\langle e_i, e_j\rangle)$. Then we define the determinant $\det L$ of L by the deter-

*) This paper is a part of author's master thesis at Tohoku University.