Invariance of strong paracompactness under closed-and-open maps

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Abstract: In this paper we strengthen a result of Ponomarev [6] on the invariance of Strong Paracompactness under open perfect maps. Namely, we prove that Strong Paracompactness is invariant under closed-and-open maps. This result follows after we give four new characterizations of strong paracompactness.

Key words: Strong paracompactness; closed-and-open maps.

§1 Preliminaries. In this paper by a space we mean a Hausdorff topological space and by a map, a continuous map of spaces.

Strongly paracompact spaces were defined by Dowker [2]. Unlike most covering properties, strongly paracompact spaces are not invariant under perfect maps [1]. This can be seen from the following result by Ponomarev [5].

Theorem 1.1. *Every paracompact space is the image of a strongly paracompact space under a perfect map.* □

Ponomarev [6] has shown that strong paracompactness is an invariant of open perfect maps. Indeed, Ponomarev [7] strengthened this result by showing that strong paracompactness is preserved under Λ-maps in the realm of regular spaces, where a map $f: X \to Y$ is called an Λ-map if it satisfies the following two conditions : (a) the image of every clopen (closed-and-open) set is a clopen set and (b) for every point $y \in Y$ and every cover $\mathcal{U}_y$ of the set $f^{-1}y$ by clopen sets admits a finite subcover. The aim of this paper is to show that in fact in the realm of regular spaces, strong paracompactness is preserved under CO-maps, that is maps satisfying condition (a) above. As a consequence we get that strong paracompactness is an invariant of closed-and-open maps (i.e. maps which are both closed and open).

Let $\mathcal{P} = \{P_\alpha : \alpha \in \mathcal{A}\}$ be a collection of subsets of a set X. By a *chain* from P_α to $P_{\alpha'}$ we mean a finite sequence $P_{\alpha(1)}, P_{\alpha(2)}, \ldots, P_{\alpha(k)}$ of elements of $\mathcal{P}$ such that $\alpha(1) = \alpha$, $\alpha(k) = \alpha'$ and $P_{\alpha(i)} \cap P_{\alpha(i+1)} \neq \emptyset$ for $i = 1, \ldots, k-1$. The collection $\mathcal{P}$ is said to be *connected* if for every pair $P_\alpha, P_{\alpha'}$ of elements of $\mathcal{P}$ there exists a chain from P_α to $P_{\alpha'}$. For every collection $\mathcal{P}$ the *components* of $\mathcal{P}$ are defined as maximal connected subcollections of $\mathcal{P}$, that is connected subcollections of $\mathcal{P}$ which are not proper subsets of any connected subcollection of $\mathcal{P}$.

Remember that a collection $\mathcal{P}$ of subsets of a set X is said to be star-finite (star-countable) if for every $P \in \mathcal{P}$ the collection $\{Q \in \mathcal{P} : Q \cap P \neq \emptyset\}$ is finite (countable). A space X is called *strongly paracompact* (otherwise called *hypocompact*) if every open cover of X has a star-finite open refinement. Thus every strongly paracompact space is paracompact but the converse is not true. In fact there exist metric spaces which are not strongly paracompact [1].

The following lemma will be used below (see for example [1] or [3]).

Lemma 1.2.

(1) *Every collection $\mathcal{P}$ of subsets of a set X decomposes into the union of its components.*

(2) *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct components of $\mathcal{P}$, then $(\cup \mathcal{P}_1) \cap (\cup \mathcal{P}_2) = \emptyset$.*

(3) *If $\mathcal{P}$ is star-countable, then each component is a countable subcollection of $\mathcal{P}$.* □

For a collection $\mathcal{P}$ of subsets of a set X and an infinite ordinal number τ let $\mathcal{P}^\tau = \{\cup \mathcal{Q} : \mathcal{Q} \subset \mathcal{P}, |\mathcal{Q}| < \tau\}$. The collection $\mathcal{P}^\omega = \mathcal{P}^{\omega_0}$ is usually denoted by $\mathcal{P}^F$. In this paper we will be interested in the particular case of $\tau = \omega_1$, the first uncountable ordinal. Thus $\mathcal{P}^{\omega_1}$ is the collection of all unions of at most countable subcollections from $\mathcal{P}$.

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