Exotic group actions in dimension four and Seiberg-Witten theory

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Topology of smooth 4-manifolds has been studied extensively by Donaldson and Seiberg-Witten theory. In [10] we used Donaldson invariants of degree 0 to give examples of exotic free actions of certain finite groups in dimension 4. In this paper we will generalize the result in [10] by Seiberg-Witten theory. We discuss Donaldson and Seiberg-Witten invariants for connected sums of 4-manifolds and rational homology 4-spheres in §1 according to [11]. In §2 by the constructions similar to those in [10] together with Cooper-Long's result [1] we show

Theorem. *For any nontrivial finite group G there exists a 4-manifold that has infinitely many free G actions so that their orbit spaces are homeomorphic but mutually non-diffeomorphic.*

§1. Invariants for some reducible manifolds. Let us recall the definitions of Donaldson and Seiberg-Witten invariants briefly. See [2], [6], [8], [12] for details. Let X be a closed smooth oriented 4-manifold with $b_1(X) = 0$, $b_2^+(X) > 1$ and let P be a principal $SO(3)$ bundle over X with $w_2(P) \equiv w \pmod 2$ for some $w \in H^2(X, \mathbf{Z})$ (and hence P is a reduction of a $U(2)$ bundle $\tilde{P}$). Hereafter $w \pmod 2$ is denoted simply by w. Let $\mathcal{G}_P$ be the set of automorphisms of P covered by those of $\tilde{P}$ with $\det = 1$. Define $\mathcal{M}_P$ to be the space of ASD (anti-self-dual) connections modulo $\mathcal{G}_P$ with respect to a generic metric on X. Then for the symmetric product $z = x^t v_1 \cdots v_s$ with the generator x of $H_0(X)$ and $v_i \in H_2(X)$, there exists a subspace $\mathcal{M}_P \cap V_z$ of codimension $4t + 2s$ in $\mathcal{M}_P$ such that the Donaldson invariant $D_X^w(z)$ is defined by the number of points in $\mathcal{M}_P \cap V_z$ counted with sign for a bundle P with $w_2(P) \equiv w$ and $-2p_1(P) - 3(1 + b_2^+(X)) = 4t + 2s$ (put $D_X^w(z) = 0$ if there does not exist such a bundle). Here note that if there are no flat connections on any $SO(3)$ bundle over X with $w_2 \equiv w$ then $\mathcal{M}_P \cap V_z$ is compact ([6]). Otherwise to avoid the flat connections we replace (X, P) by $(X \# \overline{CP}^2, P \# Q)$, where Q is the reducible $SO(3)$ bundle over $\overline{CP}^2$ with w_2 being the Poincare dual of the generator z_0 of $H_2(\overline{CP}^2, \mathbf{Z})$ modulo 2, and replace $D_X^w(z)$ by $D_{X\#\overline{CP}^2}^{w+z_0}(zz_0)$ (Morgan-Mrowka trick, [6]). In Seiberg-Witten theory, we consider a spin^c structure c on X, the associated $\pm$ spinor bundle $W^{\pm}$, and its determinant complex line bundle L over X. Then the Seiberg-Witten moduli space $\mathcal{M}_X(c)$ is the space of pairs of connections A on L and cross sections ϕ of W^+ satisfying the Seiberg-Witten equation modulo $\mathrm{Map}(X, S^1)$.

$$(SW)\quad \mathcal{D}_A(\phi) = 0,\ F^+(A) + \delta = (\phi^* \otimes \phi)_0$$

(see [8], [12] for the definitions.) The space $\mathcal{M}_X(c)$ is a compact oriented manifold of dimension $d(L) = (c_1(L)^2 - 2\chi - 3\sigma)/4$ for a generic metric on X where χ and σ are the euler number and the signature of X. Hereafter $c_1(L)$ is denoted simply by L. The Seiberg-Witten (SW) invariants $SW_X(L)$ for L with $d(L) = 0$ is the sum of the numbers of points in $\mathcal{M}_X(c)$ counted with sign for all spin^c structures c corresponding to L. (see [8] for the definition in case $d(L) > 0$.) L is called a Seiberg-Witten (SW) class if $SW_X(L) \neq 0$. X is called SW simple if $SW_X(L) = 0$ whenever $d(L) > 0$. Hereafter we assume that $H_1(X, \mathbf{Z}) = 0$, $b_2^+(X) > 1$, and Y is a rational homology 4-sphere. Moreover we assume that X is SW simple and KM simple, that is, $D_X^w(x^2 z) = 4D_X^w(z)$ for any $w \in H^2(X, \mathbf{Z})$, $z \in Sym(H_0(X) \oplus H_2(X))$, and satisfies the following equation discussed in [12].

$$(W)\quad D_X^w((1 + x/2)e^v) = 2^{2+(7\chi+11\sigma)/4} e^{Q/2} \sum (-1)^{(w^2+wL)/2} SW_X(L) e^L(v)$$

where $v \in H_2(X)$, Q is the intersection form of X, and the sum on the right hand side is taken over all the SW classes L of X.

The following results about these invariants for $X \# Y$ may be known to the experts, but we cannot find them in explicit forms in the literature.

Proposition 1.1 [11]. *If X satisfies the above*