A special divisor on a double covering of a compact Riemann surface

By Shigeo MITSUNARI

Research Institute for Mathematical Sciences, Kyoto University

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Introduction. By a curve we shall mean a connected compact Riemann surface. Let $l(D) := \dim H^0(C, \mathcal{O}(D))$ and $c(D) := \deg D - 2l(D) + 2$ for a divisor D on a curve C of genus $g \geq 2$. It is easy to check $c(D) = c(K_C - D)$ by using the Riemann-Roch theorem. Here K_C is a canonical divisor of C. Clifford's theorem states that $c(D) \geq 0$ if $l(D) > 0$ and $l(K_C - D) > 0$; moreover if there is a divisor D such that $l(D) > 0$, $c(D) = 0$ but $D \not\sim 0$ and $D \not\sim K_C$, then C is a hyperelliptic curve. In other words, we can say that a curve has a special divisor D with $c(D) = 0$ if and only if it is 0-hyperelliptic, where a special divisor D means $2 \leq \deg D \leq g - 1$ and $l(D) > 0$ (so $l(K_C - D) > 0$), a g'-hyperelliptic curve means a curve which is a double covering of a curve of genus g'.

We would like to classify double coverings of a curve with small genus by the index cliff $(C) := \min\{c(D) : D$ is a special divisor on C, $l(D) \geq 2\}$. We show that a curve having a special divisor D with small $c(D)$ is g'-hyperelliptic with $g' \leq c(D)/2$ [Theorem 1], and conversely, a g'-hyperelliptic curve has a special divisor D with $c(D) = 2g'$ [Theorem 2]. In particular, we obtain a necessary and sufficient condition for a curve to be 1-hyperelliptic [Corollary].

Main results. Theorem 1. *Let C be a curve of genus $g \geq 2$. Assume that there is an effective base-point-free divisor D with deg $D \leq g - 1$, $l(D) \geq c(D) + 3$. Then $c(D)$ is even and C is a g'-hyperelliptic curve with some $g' \leq c(D)/2$, $g \geq 6g' + 5$.*

Proof. To give a proof of this theorem, we use the following inequality of Castelnuovo [1] (p.116):

Lemma. *Let C' be a curve that admits a birational mapping onto a (not necessarily smooth) non-degenerate curve (i.e., a curve not contained in any hyperplane of the projective n-space $\mathbf{P}^n$) of degree d' in $\mathbf{P}^n$. Then the genus of C' satisfies the inequality $g(C') \leq m(m-1)(n-1)/2 + m\varepsilon$, where $m := [(d'-1)/(n-1)]$ and $\varepsilon := (d'-1) - m(n-1)$.*

Under the hypothesis of Theorem 1, $c := c(D) \geq 0$ by Clifford's theorem. Since D is base-point-free, we can define a map $\varphi : C \to \mathbf{P}^n$ associated with D. $\varphi(C)$ is non-degenerate by construction. Let C' be the normalization of $\varphi(C)$, $\nu : C' \to \varphi(C)$ the normalization map and $\tilde{\varphi} : C \to C'$ the induced map of φ. Put $d := \deg D$, $n := l(D) - 1$, $g' := g(C')$ and $d' := \deg \varphi(C)$. Then $c = d - 2n$ and $n \geq c + 2$.

Claim. $\deg \varphi = 2$.

1. If $\deg \varphi \geq 3$, then $d' \leq d/3$ and $n - d' \geq n - d/3 = (n-c)/3 > 0$. The above lemma implies $g = 0$ so $C' = \mathbf{P}^1$. Put $\mathcal{O}_{\mathbf{P}^1}(N) := \nu^*\mathcal{O}_{\mathbf{P}^n}(1)$. Since $\varphi(C)$ is nondegenerate, $\nu^* : \Gamma(\mathbf{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbf{P}^1, \mathcal{O}(N))$ is injective, so we get $n \leq N$. Since $\mathcal{O}(D) = \tilde{\varphi}^*\mathcal{O}(N)$, $d = N \deg \tilde{\varphi} = N \deg \varphi$. Therefore $3 \leq \deg \varphi = d/N \leq d/n < 3$, which is impossible.
2. If $\deg \varphi = 1$, then $d' = d$, $g' = g$ and $m = [(d'-1)/(n-1)] = [2 + (c+1)/(n-1)]$.
   (a) If $n \geq c + 3$, then $m = 2$, $\varepsilon = c + 1$ and
   $g \leq n - 1 + 2(c+1)$ (*by Castelnuovo*)
   $= 2d - 3n + 1$ (*by* $c = d - 2n$)
   $\leq d - 2$ (*by* $3n \geq d + 3$).
   This contradicts $d \leq g - 1$.
   (b) If $n = c + 2$, then $m = 3$, $\varepsilon = 0$ and $g \leq 3(n-1) = d - 1$, which also conflicts.

As a consequence we get $\deg \varphi = 2$, so $\tilde{\varphi}$ is a double covering map. Therefore $d' = d/2$; hence d and c are even. Again using Castelnuovo's lemma, we get $g' \leq d' - n = c/2$. Since $g - 1 \geq d = c + 2n \geq 3c + 4 \geq 6g' + 4$, Theorem 1 is proved. Q.E.D.

Proposition. *In the above notation, let σ be the involution of C compatible with $\tilde{\varphi}$. Then D is invariant under the action of σ^*.*

Proof. If x in the support of D is not a