An inequality among infinitesimal characters related to the lowest K-types of discrete series

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(Communicated by Kiyosi ITÔ, M. J. A., April 13, 1998)

1. Introduction. Let G be a connected real semi-simple Lie group with finite center, and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Let $G = KAN$ be an Iwasawa decomposition of G and M a centralizer of A in K.

A famous Parthasarathy's Dirac operator inequality in [7] (see also, [1,5,8]) asserts that the length of the highest weight of a representation of $\mathfrak{k}$ occuring in the Harish-Chandra module of an irreducible unitary representation of G must be at least the eigenvalue of the Casimir operator of $\mathfrak{g}$.

In the present note, we shall give some inequality for the infinitesimal characters of irreducible representations of M. This inequality resembles the Dirac operator inequality in character. In fact, it relates to the discrete series of G via a lowest K-type.

Since the group M is, in general, considerably small in K it seems hard to expect any inequalities among characters of representations of M which are obtained by the restriction of representations of K. Moreover, a proper meaning of the group M is somewhat mysterious although its structural definition is clear. In this sense, it is important to ask roles of M from various point of view. This is the aim of the study on a comparison among representations of M. In fact, we shall show that a "length" of the *dominant M-type* (see §2 for the precise definition) of the lowest K-type of a discrete series of G dominates all the other such dominant M-types which appear in a Weyl group-"orbit" of the lowest K-type. The inequality may have also a possibility to provide an information about a "scale" of parameters among various embedding of discrete series into non-unitary principal series induced from a minimal parabolic subgroup MAN. We shall lastly propose some questions concerning the inequality.

The author would like to express his thanks to Professor H. Ochiai for his valuable comments.

2. Statement and proof. Assume that rank $(K) = \mathrm{rank}(G)$. Then G contains a Cartan subgroup T which lies in K. We denote by $\mathfrak{t}$ the Lie algebra of T. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Note that $\mathfrak{t} \subset \mathfrak{k}$. Let $(\ ,\ ) = B|_{\mathfrak{p}\times\mathfrak{p}}$, where B is the Cartan-Killing form of $\mathfrak{g}$. For any subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ we denote by $\mathfrak{l}_{\mathbf{C}}$ the complexification of $\mathfrak{l}$. Let $\Delta = \Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ be the set of non-zero roots of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ and let $\Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{n}}$ be the set of compact, noncompact roots respectively, i.e. $\Delta_{\mathfrak{k}} = \Delta_{\mathfrak{k}}(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ and $\Delta_{\mathfrak{n}} = \Delta \setminus \Delta_{\mathfrak{k}}$. Let $W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ be a Weyl group for the root system Δ. Let $\mathrm{ad}: \mathfrak{k} \to SO(\mathfrak{p}, (\ ,\ ))$ be the adjoint representation. If (σ, S) is the spin representation of $SO(\mathfrak{p}, (\ ,\ ))$ defined through the Clliford algebra of $\mathfrak{p}$, let L be the composition of σ and $\mathrm{ad}\,\mathfrak{k}|_{\mathfrak{p}}$. Since the dimension of $\mathfrak{p}$ is always even, we have an irreducible decomposition of σ; $\sigma = \sigma^+ \oplus \sigma^-$, where $\sigma^{\pm}$ are the half-spin representations. Set $L^{\pm} = \sigma^{\pm} \circ \mathrm{ad}$. For any choice of positive roots $\Delta^+ \subset \Delta$ put $\Delta_{\mathfrak{k}}^+ = \Delta^+ \cap \Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{n}}^+ = \Delta^+ \cap \Delta_{\mathfrak{n}}$, $2\delta = \langle \Delta^+ \rangle$, $2\delta_{\mathfrak{k}} = \langle \Delta_{\mathfrak{k}}^+ \rangle$ and $2\delta_{\mathfrak{n}} = \langle \Delta_{\mathfrak{n}}^+ \rangle$, where generally we write $\langle \Phi \rangle = \sum_{\alpha\in\Phi} \alpha$ for each subset $\Phi \subset \Delta$. Then it is known that the weights of (L, S) are of the form $\delta_{\mathfrak{n}} - \langle Q \rangle$, where $Q \subset \Delta_{\mathfrak{n}}^+$. We fix (L^+, S^+) so that $\delta_{\mathfrak{n}}$ is a weight of L^+. If $\lambda \in \mathfrak{t}_{\mathbf{C}}^*$ is a $\Delta_{\mathfrak{k}}^+$-dominant integral weight, τ_λ will denote the irreducible representation of $\mathfrak{k}_{\mathbf{C}}$ with highest weight λ. Each weight of L occurs with multiplicity one. In fact, Parthasarathy showed in [7] that

$$L^+ = \bigoplus_{\substack{s\in W^1 \\ \det s = 1}} \tau_{s\delta-\delta_{\mathfrak{k}}}, \quad L^- = \bigoplus_{\substack{s\in W^1 \\ \det s = -1}} \tau_{s\delta-\delta_{\mathfrak{k}}},$$

Dedicated to Professor Reiji Takahashi on the occasion of his seventieth birthday.

Partially supported by Grant-in-Aid for Scientific Research (B) No. 09440022, the Ministry of Education, Science, Sports and Culture of Japan.