Generating functions of the Jacobi polynomials and related Hilbert spaces of analytic functions

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1. Introduction. In the previous paper [5], we showed that a generating function of the Gegenbauer polynomials can be regarded as the integral kernel of a unitary mapping from an L^2 space onto a Hilbert space of analytic functions. Moreover, we gave in [6] a similar construction for the system of the zonal spherical functions on the homogeneous space $U(n)/U(n-1)$, which is geometrically analogous to the space $SO(n)/SO(n-1)$ whose zonal spherical functions are essentially given by the Gegenbauer polynomials. Problems of this kind were discussed first in [1]. The purpose of this paper is to show that a similar construction is also possible for the Jacobi polynomials, which are generalizations of the Gegenbauer polynomials.

Let $\boldsymbol{R}$, $\boldsymbol{C}$ be the fields of real and complex numbers, respectively. For positive numbers α and β, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n = 0, 1, 2, \cdots$, are defined by the Rodrigues formula (cf. [2]):

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$

Then the system $\{P_n^{(\alpha,\beta)}(x) ; n = 0, 1, 2, \cdots\}$ has the orthogonality relations (cf. [2]):

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx = \begin{cases} 0 & (n \neq m) \\ \dfrac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} & (n = m), \end{cases}$$

and the generating function (cf. [4]): for $-1 < x < 1$ and $z \in \boldsymbol{C}$, $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n}{(\alpha+1)_n} z^n P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)(z+1)}{(1-z)^{\alpha+\beta+2}}\, {}_2F_1\left(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \alpha+1; \frac{2z(x-1)}{(1-z)^2}\right),$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ (Γ is the Gamma function) and ${}_2F_1(a, b; c; t)$ is the Gaussian hypergeometric function. We denote by $F_{\alpha,\beta}(z, x)$ the right hand side of this formula.

Let $\varphi_n^{(\alpha,\beta)}(x)$ be the normalization of $P_n^{(\alpha,\beta)}(x)$ with respect to the inner product defined by

$$(\psi, \varphi)_{\alpha,\beta} = \int_{-1}^{1} \overline{\psi(x)}\varphi(x)(1-x)^\alpha(1+x)^\beta dx.$$

Then the system of the functions $\varphi_n^{(\alpha,\beta)}(x)$, $n = 0, 1, 2, \cdots$, is an orthonormal basis of the Hilbert space $\mathscr{L}^2_{\alpha,\beta} = L^2\Big((-1, 1), (1-x)^\alpha(1+x)^\beta\Big)$ with the inner product $(\ ,\ )_{\alpha,\beta}$.

In this paper, we shall give a Hilbert space $\mathscr{H}_{\alpha,\beta}$ of analytic functions and a unitary operator of $\mathscr{L}^2_{\alpha,\beta}$ onto $\mathscr{H}_{\alpha,\beta}$ whose integral kernel is the generating function $F_{\alpha,\beta}(z, x)$.

Suppose that α, β are positive numbers throughout this paper.

2. Hilbert space $\mathscr{H}_{\alpha,\beta}$. We define the function $\rho_{\alpha,\beta}(t)$ for $0 < t < 1$ by

$$\rho_{\alpha,\beta}(t) = t^{\frac{\alpha+\beta-1}{2}}\int_t^1 u^{-\frac{\alpha+\beta+1}{2}}(1-u)^{\beta-1}du\int_{\frac{t}{u}}^1 v^{-\frac{\beta-\alpha+1}{2}}(1-v)^{\beta-1}dv,$$

and denote by $\mathscr{H}_{\alpha,\beta}$ the Hilbert space of analytic functions on the unit open disk B in $\boldsymbol{C}$ with the inner product defined by

$$< f, g >_{\alpha,\beta} = \int_B \overline{f(z)}g(z)\rho_{\alpha,\beta}(|z|^2)dz,$$

where $dz = dxdy$, $z = x + iy$ ($x, y \in \boldsymbol{R}$). The functions $g_n(z) = z^n$, $n = 0, 1, 2, \cdots$, form an orthogonal basis in $\mathscr{H}_{\alpha,\beta}$ and the norm $\|g_n\| = \sqrt{< g_n, g_n >_{\alpha,\beta}}$ is given in the following.

Lemma 1. *For a nonnegative integer n, we have*

$$< g_n, g_n >_{\alpha,\beta} = \frac{2\pi\big(\Gamma(\beta)\big)^2}{2n+\alpha+\beta+1}\frac{\Gamma(n+1)}{\Gamma(n+\beta+1)}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)}$$

Proof. In exchanging orders of integrals, we