Orbits of triangles obtained by interior division of sides

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Abstract: Plane triangles are classified by similarity. Let Ω be the set of these equivalence classes of triangles, and $[ABC] \in \Omega$ be the class of triangles which are similar to ΔABC, Putting $x = \angle A$, $y = \angle B$, $z = \angle C$, $[ABC]$ is represented by a point in $\Pi = \{(x, y, z) \mid x + y + z = \pi, x, y, z > 0\}$. By making interior division of sides of ΔABC, we define an orbit in Π, starting from $[ABC]$. It is determined by a differentiable dynamical system, and is the intersection of Π and the surface $\cot x + \cot y + \cot z =$ const.

Key words: Triangles; interior division; convex closed curve; four-vertex theorem.

1. Introduction. We consider here the set $\boldsymbol{T}$ of all triangles on the Euclidean plane. Triangles in $\boldsymbol{T}$ are classified by similarity. In this note, we say that ΔABC is similar to $\Delta A'B'C'$ and write as $\Delta ABC \simeq \Delta A'B'C'$ if $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$. It defines an equivalency. Put

(1.1) $[ABC] = \{\Delta A'B'C' \mid \Delta A'B'C' \simeq \Delta ABC\}$

Obviously $[ABC] \cap [A'B'C'] \neq \emptyset$ if and only if $[ABC] = [A'B'C']$. We define

(1.2) $\Omega = (\boldsymbol{T}/\simeq) = \{[ABC] \mid \Delta ABC \in \boldsymbol{T}\}$.

Note that, in general, $[ABC]$, $[BCA]$, and $[CAB]$ are mutually distinct in Ω.

Write $\angle A = x$, $\angle B = y$, $\angle C = z$, then $[ABC]$ is represented as a point in $\boldsymbol{R}^3$. Ω is idenified with the set

(1.3) $\Pi = \{(x, y, z) \mid x + y + z = \pi, x > 0, y > 0, z > 0\}$.

The class of regular triangles is denoted by a point $(\pi/3, \pi/3, \pi/3)$. Points on the boundary of Π denote degenerate triangles. A point in Π corresponding to $[ABC]$ is denoted also by $[ABC]$.

Consider a triangle $\Delta ABC \in [ABC]$. On each side of it, take the point of interior division with the ratio $t : (1 - t)$, where $0 \leqq t \leqq 1$. The point on the side AB is denoted by $A(t)$. Similarly for $B(t)$ and $C(t)$ on BC and CA, respectively. Put

(1.4) $T_0(ABC) = \{[A(t)B(t)C(t)] \mid 0 \leqq t \leqq 1\}$.

$T_0(ABC)$ is represented by a continuous arc in $\Pi \subset \boldsymbol{R}^3$ which connects $[ABC]$ with $[BCA]$. Obviously $T_0(ABC) \cup T_0(BCA) \cup T_0(CAB)$ is a closed curve in Π. Since $B = A(1)$, $C = B(1)$, $A = C(1)$, we may define $[A(1+t)B(1+t)C(1+t)]$, $0 \leqq t \leqq 1$, as $[B(t)C(t)A(t)]$, $0 \leqq t \leqq 1$. Similarly $[A(2+t)B(2+t)C(2+t)]$ may be defined as $[C(t)A(t)B(t)]$. Now for any $t \in \boldsymbol{R}$, let $[t]$ be the greatest integer not exceeding t. Writing $t^* = t - [t]$, $0 \leqq t^* < 1$, we define

(1.5)
$$[A(t)B(t)C(t)] = \begin{cases} [A(t^*)B(t^*)C(t^*)], & \text{if } [t] = 3m + 0 \text{ for some integer } m, \\ [B(t^*)C(t^*)A(t^*)], & \text{if } [t] = 3m + 1 \text{ for some integer } m, \\ [C(t^*)A(t^*)B(t^*)], & \text{if } [t] = 3m + 2 \text{ for some integer } m. \end{cases}$$

For example, if $-1 < t < 0$, then $[t] = -1 = -3 + 2$ and $t^* = 1 - |t|$. Hence $[A(t)B(t)C(t)] = [C(1 - |t|)A(1 - |t|)B(1 - |t|)]$. By (1.5), we define as a continuation of (1.4),

(1.6) $T(ABC) = \{[A(t)B(t)C(t)] \mid t \in \boldsymbol{R}\}$,

which is represented by a closed curve in Π.

There are some investigations on triangles obtained by interior division of sides of ΔABC, e.g. [4]. However, as far as I know, we have almost no knowledge about the set $T(ABC)$, except the case when $t = 1/2$, where $\Delta B(1/2)C(1/2)A(1/2) \simeq \Delta ABC$.

In this note we investigate the set $T(ABC)$. Establishing some lemmas on 2×2 matrices, we will see that $T(ABC)$ is a continuously differentiable curve, and find the system of differential equations which determines the curve. It shows that $T(ABC)$ is a convex curve, represented by the intersection of Π and the surface

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