About splitting numbers

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(Communicated by Kiyosi ITÔ, M. J. A., Feb. 12, 1998)

The main purpose of this paper is to present a proof of the following theorem with some expositions for working mathematician's who are not set theorists.

Theorem 1. *Suppose that κ is an uncountable regular cardinal. Then, the splitting number of κ is strictly larger than κ if and only if κ is a weakly compact cardinal.*

In August 1992, the author reported a sketch of a proof of Theorem 1 and that of Proposition 2, in a meeting at the Research Institute of Mathematical Sciences, Kyoto University [10]. Since then, a previous version of our current paper has been circulated among some set theorists. In October 1992, Zapletal [11], who was a graduate student of the Pennsylvania State University at that time, informed us that he improved our results. Nowadays, the theory of splitting number at uncountable cardinals relates to various branches of set theory such as inner model theory and Shelah's pcf theory. Our current paper is the final version of the preprint with the same title, which appears at the list of references of Zapletal's paper [11].

It is not hard to see that our argument gives an alternative proof of the following fact due to Johnson [4, Corollary 2]: "Suppose that κ is an uncountable regular cardinal. Then, κ is weakly compact iff I_κ is WC iff I_κ is (κ, κ)-distributive". Johnson showed this fact by using forcing. In [11, Lemma 4], Zapletal cited our Theorem 1, and he presented a modified proof that uses ultrapowers. On the other hand, our proof of Theorem 1 is purely combinatorial.

§1. The classical splitting number. Not only the cardinality of the continuum, but also some invariants of the continuum have applications in studies of topological issues; the splitting number s is one of such invariants [2]. To see the definition of the splitting number, let us define some auxiliary concepts. For a set X and a cardinal number κ, the collection of all sets that satisfy the following two conditions is denoted by $[X]^\kappa$: (1) it is a subset of X; (2) it has cardinality κ. In accordance with usual manner of set theory, we identify $\aleph_0$ with $\boldsymbol{N}$, the collection of all natural numbers (including zero). In the following, ω stands for $\aleph_0 (= \boldsymbol{N})$. Moreover, ω_α $(= \aleph_\alpha)$ denotes the α-th cardinal number; $\omega_0 = \omega$ and ω_1 is the least uncountable cardinal. For a cardinal number κ, 2^κ denotes the cardinality of the power set of κ; thus, 2^ω is the cardinality of the continuum. If $\kappa = \omega_\alpha$, then κ^+ stands for $\omega_{\alpha+1}$. ZFC denotes the usual formal system of set theory i.e. Zermelo-Fraenkel set theory with axiom of choice. Jech's book [3] is a standard textbook of basic concepts of set theory.

$\mathcal{S} \subseteq [\omega]^\omega$ is called *a splitting family* if for each $X \in [\omega]^\omega$, there exists an $A \in \mathcal{S}$ such that $|X \cap A| = |X \setminus A| = \omega$, where $|X|$ denotes the cardinality of X. The *splitting number* s is the minimum cardinality of a splitting family: $s =_{def} \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega$ is a splitting family$\}$.

The second assertion of the following fact is shown by forcing.

Fact 1 ([2]).

1. $\omega_1 \leq s \leq 2^\omega$.
2. *The consistency of ZFC implies the consistency of the following: "$ZFC + \omega_2 \leq s$".*

§2. The splitting number on ω_1. In 1991, Shizuo Kamo at Osaka Prefecture University defined the splitting number on an arbitrary infinite cardinal number κ as follows. $\mathcal{S} \subseteq [\kappa]^\kappa$ is called *a splitting family on κ* if for all $X \in [\kappa]^\kappa$ there exists an $A \in \mathcal{S}$ such that $|X \cap A| = |X \setminus A| = \kappa$. *The splitting number of κ*, which is denoted by $s(\kappa)$, is the minimum cardinality of a splitting family on κ: $s(\kappa) =_{def} \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\kappa]^\kappa$ is a splitting family$\}$. Clearly, the classical splitting number s is $s(\omega)$. His graduate student Motoyoshi tried to show, under the assumption that κ is an uncountable regular cardinal, the fol-

Mathematics Subject Classifications: 03E55.